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EDITED BY

EARLE R. HEDRICK

DAVID R. CURTISS WILLIAM R. LONGLEY

WITH THE ASSISTANCE OF

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THEOPHIL H. HILDEBRANDT DERRICK N. LEHMER

JOHN R. KLINE HENRY L. RIETZ

CAROLINE E. SEELY

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SOME RECENT CONTRIBUTIONS TO ALGEBRAIC GEOMETRY*

BY VIRGIL SNYDER

1. *Introduction.* During the last decade mathematical literature has been enriched by over a thousand contributions to algebraic geometry, including about eight hundred to the narrower field of rational transformations. It would be an ambitious task to report on this immense field in one short talk. Today I wish to speak of only three problems, each one somewhat well defined, or even narrow. Of these, one furnishes a striking example of mathematical elegance in providing one solution to what was regarded as several distinct problems. The other two are capable of unlimited extension, each new enlarged field furnishing phases not existing in the earlier ones.

2. *Series of Composition of Veneroni Transformations.* The Cremona transformations determined by a system of bilinear equations between the systems of coordinates of the two associated spaces were among the first to be considered in S_2 and S_3. For the general case the configuration of fundamental and principal elements can be at once expressed in terms of the vanishing of certain determinants of a matrix. (Segre.†) This has recently been generalized to spaces of higher dimensions by various authors, in particular to S_n by Godeaux.‡ Numerous particular cases have been considered. If $x_i x_i' = x_k x_k'$, the extreme case of inversion results. Transformations made up of this inversion and of collineations have no fundamental curves of the first kind; they have a series of composition somewhat similar to that of the general case for S_2.

If in every bilinear equation the coefficients a_{ik} and a_{ki} are equal, the transformation is involutorial and can be expressed in terms of polarity as to a series of quadrics or as to null systems. In case all the polarities are quadric and the quadric primals are independent, there are only a finite number of invariant

* Presented to the Society at the symposium held in New York, March 30, 1934, at the invitation of the program committee.

† C. Segre, Rendiconti dei Lincei, (2), vol. 9 (1900), pp. 253–260.

‡ L. Godeaux, Lombardo Istituto Rendiconti, (2), vol. 43 (1910), pp. 116–119.

points. These have been discussed by M. del Re.* The case of $n=4$ has been investigated by Miss Alderton,† and various further properties have been given by Wong‡ and Roth.§

The general Cremona transformation for S_r, $r>2$, does not have a series of composition, that is, it cannot be expressed as the product of transformations of lower order.

A Cremona transformation in S_3 of order $n=4k+r$ having as base curve of maximal multiplicity a sextic curve C of genus $p=3$ and of multiplicity $k+1$, is the product of a (3, 3) and of another transformation of order n. (Piazzolla-Beloch.‖)

Similarly, a $T_{2n-1}: C_3^{(n-1)}$, the only other base-system consisting of bisecants of C_3, has a definite series of composition. (Montesano.¶) This case has been developed further by Tinto.**

3. *Case* $n=3$. The most important particular case from this point of view for $n=3$ is that in which the base C_6 is composed of four skew lines r_i and their two transversals u, v. The inverse is defined by cubic surfaces through four skew lines r_i' and their two transversals u', v'. A point on r_i has for image a line meeting three lines r_k', so that r_i itself is transformed into the quadric through three lines r_k'. A point on u (or v) has the whole line u' (or v') for image, and conversely. These transversals are simple fundamental lines of the second kind.

The image of an arbitrary line is a space cubic meeting each r_i' twice, but not meeting u' or v'. A line r meeting u and v is transformed into a line meeting u' and v'. The (1, 1) correspondence between this line and its image is a collineation, such

* Maria del Re, Naples Rendiconti, (3), vol. 28 (1922), pp. 203–211, and (3), vol. 35 (1929), pp. 208–217.

† Nina Alderton, University of California Publications in Mathematics, vol. 1 (1923), pp. 345–358.

‡ B. C. Wong, Annals of Mathematics, (2), vol. 27 (1926), pp. 330–336, and vol. 28 (1927), pp. 251–262; American Journal of Mathematics, vol. 49 (1927), pp. 383–388; this Bulletin, vol. 35 (1929), pp. 829–832.

§ L. Roth, Proceedings of the Cambridge Philosophical Society, vol. 29 (1933), pp. 178–183.

‖ M. Piazzolla-Beloch, Annali di Matematica, (3), vol. 16 (1909), pp. 27–98.

¶ D. Montesano, Naples Rendiconti, (3), vol. 27 (1921), pp. 116–127 and 164–175.

** F. J. Tinto, Proceedings of the Edinburgh Mathematical Society, vol. 34 (1916), pp. 133–145.

that the points r, u and r, v are transformed into r', u' and r', v'. Through a point P in (x) passes a single line meeting u, v; through its image P' passes a single line meeting u', v'. The collineation is now uniquely fixed. Let u be taken as $x_1=0$, $x_2=0$ and v as $x_3=0$, $x_4=0$. For r_1, r_2 the lines $x_1=0$, $x_3=0$ and $x_2=0$, $x_4=0$ may be taken. Then for r_3 take $ax_1+bx_2=0$, $cx_3+dx_4=0$ and r_4' is $a'x_1+b'x_2=0$, $c'x_3+d'x_4=0$.

Since the quadric surface through any three lines r_i contains the two transversals, a quadric and any plane of the pencil through the residual line will be a cubic surface of the homaloidal system. If $H_{13}=0$ is the equation of the quadric through all the lines except $x_1=0$, $x_3=0$, etc., we may write

$$x_1 = x_1H_{13},\quad x_2' = x_2H_{24},\quad x_3' = x_3H_{13},\quad x_4' = x_4H_{24}.$$

Similar pencils are of the forms

$$(ax_1 + bx_2)H_{a,c} + (cx_3 + dx_4)H_{a,c} = 0,$$
$$(a'x_1 + b'x_2)H_{a',c'} + \text{etc.} = 0.$$

By writing the equations of H_{13}, H_{24} we can now obtain the forms of the defining bilinear equations

$$x_1'x_3 - x_1x_3' = 0, \qquad x_2'x_4 - x_2x_4' = 0,$$
$$(cd')(ax_1 + bx_2)(b'x_3' - a'x_4')$$
$$- (ab')(d'x_1' - c'x_2')(cx_3 + dx_4) = 0.$$

4. *Products of* (3, 3) *Cremona Transformations.* Now consider the product of $T_1{:}r_iu,\ v;\ r_i'u',\ v'$ and $T_2{:}r_i'u',\ v';\ r_i'',\ u'',\ v''$ in which the same lines u', v' appear in T_1^{-1} and in T_2. The lines r_i', r_i' may all be distinct or in part the same. Similarly for the continued product $T_1, T_2, \cdots, T_k$. For every value of k the transformation is determined by two lines u, v of the same multiplicity and of a number of lines, of various multiplicities, meeting both of them, and of no other base elements.

Conversely, consider a transformation T defined by the two properties:

(a) To a line r meeting two skew lines u, v shall correspond a line r' meeting two skew lines u', v'.

(b) The projectivity established between r and r' shall never be degenerate, and the point r, u shall correspond to the point r', u'; similarly for r, v and r', v'.

The transformations determined by these conditions have u, v to the same multiplicity ν; a number of lines meeting both are base elements of the web, and there can be no other base elements. The transformation is of order $2\nu+1$. We may then write

$$|S_2| \sim |\phi_{2\nu}\,\phi_{2\nu+1}| : u^{\nu} v^{\nu} r_1^{\rho_1} \cdots r_p^{\rho_p},$$
$$|S_1| \sim |C_{2\nu} C_{2\nu+1}| : r_1^{2\rho_1} \cdots r_p^{2\rho_p}.$$

A plane through u (or v) is transformed into a ruled surface $R_{2\nu+1}: u'^{\nu} v'^{\nu+1} r_1'^{2\rho_1} \cdots r_p'^{2\rho_p}$.

Each line r_i is transformed into a ruled surface; each point of $r_i \sim$ a rational curve o' of order ρ_i', not meeting u', v', which generates $R_{2\rho_i}: u'^{\rho_i'} v'^{\rho_i'}$.

An arbitrary plane ω meets u, v each in a point and contains the line l joining them. Let ω' be any plane (not containing u', v') through the image of this line. Given any point T in ω, through it passes a unique line c meeting u, v. The image line c' meets u', v' and pierces ω' in T'. Between ω, ω' exists a plane Cremona transformation. Let T describe a line p in ω. The lines u, v, p determine a quadric surface, the image of which is a surface of order $4\nu+2$. The section by ω' contains l' and the image of p, hence

$$c_1 \sim c'_{4\nu+1}: U'^{2\nu+1} V'^{2\nu+1} \sum_{i=1}^{p'} R_i^{\rho_i'}.$$

Since in a plane Cremona transformation the number of F points in the two planes is the same, it follows that $p = p'$.

From the two fundamental relations of a plane Cremona transformation, we have further $\sum \rho_i = 4$, $\sum \rho_i^2 = 2\nu(\nu+1)$. Let $\rho_1 \geqq \rho_2 \geqq \cdots \geqq \rho_p$. Then

$$\rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4(\rho_4 + \cdots + \rho_p) \geqq 2\nu(\nu+1),$$
$$\rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4(4\nu - \rho_1 - \rho_2 - \rho_3) \geqq 2\nu(\nu+1),$$
$$\nu(\rho_1 - \rho_4) + \nu(\rho_2 - \rho_4) + \nu(\rho_3 - \rho_4) + 4\nu\rho_4 \geqq 2\nu(\nu+1),$$
$$\rho_1 + \rho_2 + \rho_3 + \rho_4 \geqq 2(\nu+1).$$

From this last equation it follows that if the web of surfaces, homaloids of the planes of space, is transformed by means of the (3, 3) transformation defined by r_1, r_2, r_3, r_4, the order of the surfaces of the web will be lowered. By continuing this process it follows that every transformation of this kind can be expressed as the product of (3, 3) transformations defined by four skew

base lines. (Montesano.*) The complete table of characteristics can now be constructed, and all the details of incidence determined exactly as for a plane Cremona transformation.

5. *Case* $n=4$. In S_4, consider five planes π_i, no two lying in an S_3. The system $|\phi_3^4|$ of quartic primals passing through them form a system ∞^4 of homaloids. The base M_2^{10} consists of the five planes π_i and of a ruled surface R_5 formed by the ∞^1 lines which meet all five planes. Lines meeting any four of the planes π_i are all P lines. They generate a Segre 10 nodal cubic primal V_3^3, $\psi_i : 4\pi_k$, not π_i. This cubic primal and any S_3 through π_i together form a quartic primal of the homaloidal system. The equations of the associated Cremona transformation therefore become $x_i' = \psi_i S_{3,i} : \pi_i$. Each ψ_i contains ∞^2 lines, images of the points of π_i. The inverse system is of the same form as the given one. Any two of the base planes, as π_1, π_2, meet in a point π_{12}. The plane through π_{12}, π_{23}, π_{31} meets π_1, π_2, π_3 each in a line lying on ψ_4 and ψ_5. This plane is transformed into the point P_{45}' common to π_4', π_5'; it is a double point on ψ_1', ψ_2', ψ_3'.

The primals ψ_4, ψ_5 intersect in π_1, π_2, π_3 and the plane π_{45} determined by π_{12}, π_{23}, π_{31}, and R_5. A section by any S_3 consists of three skew lines, a line meeting all of them, and of a quintic curve which is therefore elliptic. *The base* R_5 *is an elliptic ruled surface.* Every generator of R_5 is a base line and a P line having a line in (x') for image in such a way that any point on either line has the whole other line for image, $R_5 \sim R_5'$.

Two primals ϕ of the homaloidal system intersect in a surface of order 16, consisting of $5\pi_i$, R_5, and a variable surface f of order 6. A section of f by an S_3 is a sextic curve C_1 of $p=3$, meeting R_5 in 5 points. The surface f meets each π_i in an elliptic cubic curve, and R_5 in 5 generators. Three primals ϕ not belonging to the same pencil meet in a normal C_4 which has three points in each base π_i. Since a line in (x) is uniquely fixed by two points on it, it follows that a unique C_4 can be passed through two points and have five given planes of S_4 for trisecants. (Veneroni,† White,‡ Todd.§) Consider the ∞^6 lines of

* D. Montesano, Naples Atti, (2), vol. 18, (1930), 44 pp.

† E. Veneroni, Lombardo Istituto Rendiconti, vol. 34 (1901), pp. 640–644.

‡ F. P. White, Proceedings of the Cambridge Philosophical Society, vol. 21 (1922), pp. 216–227.

§ A. J. Todd, ibid., vol. 26 (1930), pp. 323–333.

S_4; $\infty^6 \,|C_4|$ have 5 given planes for trisecants. Of these, ∞^3 pass through a given point, hence a finite number have a sixth plane for trisecant, images of lines meeting f (image of the sixth plane) in three points. This number is one (White*); hence a normal C_4 can be passed through a given point, having six given planes for trisecants. It was seen that R_5 is cut by S_5 in a C_5, $p=1$; hence an S_3 through a generator g of R_5 will contain an elliptic C_4, and an S_3 through two generators meets R_5 in an elliptic cubic, hence a plane curve. There are ∞^2 pairs of generators on R_5, but if π is a plane containing a cubic curve γ_3 on R_5, through π can be passed $\infty^1 |S_3|$, each containing a pair of generators of R_5; hence R_5 *contains* ∞^1 *plane cubic curves* γ_3. The base planes π_i are among the planes π containing curves γ_3. Any plane π meeting R_5 in γ_3 is transformed into a plane π', since through every point of γ_3 passes a generator g of R_5; hence R_5' is a component of the image f' of this plane.

Let π, π' be a pair of associated planes in $T_{4,4}$. Each contains a cubic curve γ_3, γ_3' and these curves are projectively equivalent. A line l in π meets γ_3 in three points, hence its image C_4' consists of three generators of R_5' and of a line l' in π'.

The (1, 1) *correspondence between the points of two associated planes* π, π' *is a collineation.*

Now consider the homaloidal system defined by $\pi_1 \cdots \pi_5 R_5$ in (x), and its inverse $\pi_1' \cdots \pi_5' R_5'$ in (x') and a second one in (x') having R_5' as before, but a system of transversal planes $\bar{\pi}_i$, which may be distinct from π_i' or partly identical, and its inverse $\pi_1'' \cdots \pi_5'' R_5''$ in (x''). By means of the product of the associated Cremona transformations, a plane π meeting R_5 in γ_3 is transformed into a plane π'' meeting R_5'' in a γ_3'', and the (1, 1) correspondence between these planes is a collineation. Similarly for any number of components. The resulting product will always be of the form

$$T \qquad S_3 \sim \phi_{2r+1} : R_5^{r} \pi_1^{\rho_1} \cdots \pi_p^{\rho_p},$$

and the inverse will be of the same form.

Now the question arises whether every Cremona transformation of this form in S_4 can be expressed as the product of $T_{4,4}$ of the type just considered. An arbitrary S_3 meets R_5 in C_5, $p=1$; the trisecants of C_5 form a ruled surface σ_5 of order 5,

* F. P. White, loc. cit.

having C_5 for complete double curve. The image of each of these generating trisecants is a line, and these image lines describe a ruled surface σ_5' having a double curve C_5' in S_3' (not the image of S_3 in $T_{4,4}$). A (1, 1) correspondence between the points of S_3, S_3' can be established as follows: from any point P in S_3 can be drawn one and only one trisecant of R_5. (White.*) If P is not on σ_5, its associated trisecant t is not in S_3. The image t' of t meets S' in P'. Let P describe a line p in S_3. It will meet 5 lines of σ_5. Construct the ruled surface of lines t for each position of P on p, get its transform in T, then cut the transform by S_3'. The section will consist of the image of p, and of five generators of σ_5'.

Since p is a simple directrix line on the ruled surface of trisecants, its order is one greater than the number of generators in any S_3 through p. Since any S_3 through p meets R_5 in C_5 and contains a surface σ_5 of trisecants, the surface is of order 6. Its image in T is therefore of order $6(3\nu+1)$ and in S_3' the image of p is a curve of order $18\nu+1$, containing C_5' to multiplicity 6ν and lines $\omega_1, \cdots, \omega_p$ to multiplicities $6\rho_i$:

$$p \sim C_{18\nu+1}{:}C_5'^{6}\,\omega_1^{\rho_1'}\cdots\omega_{p'}^{\rho_{p'}'}.$$

For $\nu=1$ this is a particular case of the transformation belonging to a tetrahedral complex. The C_{10} of $p=11$ of the linear complex is replaced by C_5, $p=1$, and 5 of its trisecants. (Sharpe and Snyder.†)

By repeating the argument already used, and using the relations analogous to those used by Montesano,‡ the condition $\rho_1+\cdots+\rho_5 \geqq 3(\nu+1)$ can be established; hence by using the $T_{4,4}$ determined by those π_i of highest multiplicity, the order of the primals of the homaloidal net can be reduced. The transformation T can be expressed in terms of $T_{4,4}, \cdots$.

An S_3 meets ϕ_i, ϕ_k of the system in quartic surfaces through C_5 and 5 of its trisecants. The residual intersection is C_6, $p=3$, meeting C_5 in 5 points and the 5 trisecants each in three. Hence the surface f thus meets each of the five plane π_i in cubic curves, and R_5 in a composite quintic curve consisting of 5 generators.

* F. P. White, loc. cit.

† F. R. Sharpe and V. Snyder, Transactions of this Society, vol. 21 (1920), pp. 52–78.

‡ D. Montesano, Naples Rendiconti, loc. cit.

Two surfaces f meet in 36 points. The point π_{ij} is double for each primal, hence quadruple on their surface of intersection. Of these, three are accounted for by the fact that π_{ij} lies on π_i, π_j, R_5; hence f passes through π_{ij} simply.

The cubic curve in which f_6 meets π_i passes through the four points π_{ij} which lie in this plane, so that two such cubics have five intersections at points not coinciding with π_{ij}. Two f's, then, have 10 π_{ij}, 5 other points in each of the planes π_i, and one variable intersection.

A plane ω meets R_5 in 5 points, and meets each ψ_i in a cubic curve c_3 belonging to the pencil $4\pi_{ik}5\Gamma$. The image f_6' of ω contains 5 lines g' of R_5' and meets π_i' (image of ψ_i) in a cubic curve c_3', in (1, 1) correspondence with c_3. The lines g' pierce π_i' in 5 points Γ', all on c_3'. But these points also lie on γ_3', section of R_5' by π_i'. Through the five lines $g_1', \cdots, g_5'$, in which f_6' meets R_5', pass ∞^1 surfaces f_6'.

The five points G' in which any f_6' meets γ_3' in π_i' apart from the points π_{ik} are base points of a pencil of cubic curves. Let ω meet R_5 in 5 points G. Through each point G passes a generator g_i of R_5. These lines Gg_i are met by ∞^1 planes ω. Two such planes have for images surfaces f_6' having 10 points π_{ik} and 5 lines g' in common.

6. *Case $n=5$.* Let $\sigma_1, \cdots, \sigma_6$ be six three-spaces in $[5]$. They determine a homaloidal system of order 5. Any five of the σ_i, as $\sigma_1, \cdots, \sigma_5$, determine a quartic primal ψ_4, and similarly for five others. These are images of the base system σ_i' in another $[5]$. Consider $\sigma_1, \cdots, \sigma_4$ common to ψ_5, ψ_6. Any two of them intersect in a line, thus defining six lines l_{ik}, three skew lines in each of the spaces $\sigma_1, \cdots, \sigma_4$. The lines are all double lines on the primals of the system; the $[3]$ determined by σ_{12}, σ_{34} lies entirely on ψ_5, ψ_6 and similarly for the other two $[3]$ determined respectively by σ_{13}, σ_{24}; σ_{14}, σ_{23}. The section of ψ_5, ψ_6 consists of the four spaces $\sigma_1, \cdots, \sigma_4$, the three transversals just described, and a residual ruled variety of order nine, common to all the homaloids of the system.

The base of the ∞^5 homaloidal system is a three-way variety of order 9, locus of the ∞^2 lines that meet all six three-spaces of the bases σ_i.

7. *Case n General.* This system is always included among those

determined by a system of bilinear equations; hence all the configurations are expressed by the vanishing of determinants of a matrix. We may write

$$\sum a_{ik}^{(r)} x_i x_k' = 0, \qquad (r = 1, \cdots, n).$$

If $\sum_{i=1}^{n+1} a_{i,k}^{(r)} x_i = A_k^{(r)}$, then $\sum A_k^{(r)} x_k' = 0$, and the birational transformation is expressed by $x_p' = \|A_k^{(r)}\|$, $(k \neq p)$, and similarly $x_p = \|A_k'^{(r)}\|$, $(k \neq p)$. The primals $\|A_k^{(r)}\|$ are of order n, and have in common a manifold M of dimensionality $n-2$ and of order $n(n+1)/2$, consisting of the $n+1$ spaces $[n-2]$ and of a ruled variety of order $(n+1)(n-2)/2$.

The formula given by Veneroni* and repeated in the Encyklopädie (vol. III, Chapter 7, p. 967) does not give the base, but the entire intersection. For further general properties, see M. Mikam.†

Any primal of order n through M belongs to the system. The transversals of the primals in pairs include $(n-2)(n-3)/2$ spaces analogous to l_{12}, l_{24}, etc.

An $[n-r]$ of $[n]$ defined by r n-primals of $[n]$ is transformed into a manifold of dimensionality $n-r$ of order ${}_nC_{n-r}$, having with M a manifold N of dimensionality $n-r-1$ and of order $(r(n+1)/(n-r)_nC_{n-r-1}$ including the $n+1$ intersections with the base. The lines which meet the composite M in n points generate a ruled primal of order n^2-1, the Jacobian of the Cremona transformation. No line meets M in more than n points. (Segre,‡ Godeaux.§)

8. *Related Questions.* A number of questions closely related to those here considered have been solved recently. Thus, the algebraic surfaces on a cubic primal of S_4 (Archbald‖); octavic normal surfaces with a double line in $[5]$ (Babbage¶); isolated singular points in the theory of algebraic surfaces (Babbage**); some chains of theorems derived by successive projection

* Loc. cit.

† M. Mikam, Prag Ceske Akademie Rozpravy, vol. 39 (1929), No. 21.

‡ C. Segre, loc. cit.

§ L. Godeaux, loc. cit.

‖ J. W. Archbald, Proceedings of the Cambridge Philosophical Society, vol. 27 (1931), pp. 405–406, and vol. 29 (1933), pp. 484–486.

¶ D. W. Babbage, ibid., vol. 29 (1933), pp. 95–102 and 405–406.

** D. W. Babbage, ibid., vol. 29 (1933), pp. 212–230.

(Lob*). Segre's theorem concerning curves on ruled surfaces has been considered and extended by Welchman.† A useful scheme for mapping spaces S_k of an S_m upon the points of a third of order $(m-k)(k+1)$ is given by Room,‡ and Babbage gives a helpful discussion of the transformations of certain surfaces having special singularities. Welchman§ has generalized certain incidence formulas to apply to incidence scrolls, ruled surfaces generated by certain directrix spaces. It would be well to extend this investigation to apply to varieties generated by planes, [3], by similar methods. (Welchman.‖)

A discussion of the forms and possible transformations of isolated singular points of algebraic surfaces, extending the work of Godeaux,¶ was made by Babbage.** Roth†† gives further examples of surfaces in hyperspace containing triple curves and obtains some further results concerning composite surfaces contained within a given system in higher spaces.‡‡

Six lines of S_3 can have various relations to each other discussed by Todd,§§ generalizing Ursell's‖‖ study of seven lines on a quartic surface, the results of Richmond,¶¶ and the earlier paper by Wakeford.***

A detailed analytic account of the ∞^{n-2} lines generating the V_{n-1} in $[n]$ is given by Eiesland.††† These manifolds are the

* H. Lob, Proceedings of the Cambridge Philosophical Society, vol. 29 (1933), pp. 45–51.

† W. G. Welchman, ibid., vol. 29 (1933), pp. 382–388.

‡ T. G. Room, ibid., vol. 29 (1933), pp. 331–336.

§ W. G. Welchman, ibid., vol. 29 (1933), pp. 235–254.

‖ W. G. Welchman, Proceedings of the Cambridge Philosophical Society, vol. 29 (1933), pp. 103–115.

¶ L. Godeaux, Bulletin de l'Académie Royale de Belgique, Classe des Sciences, (5), vol. 15 (1929), pp. 317–318.

** Loc. cit.

†† L. Roth, Proceedings of the London Mathematical Society, (2), vol. 30 (1929), pp. 118–126; Proceedings of the Cambridge Philosophical Society, vol. 28 (1932), pp. 300–310, and vol. 29 (1933), pp. 178–183.

‡‡ L. Roth, ibid., vol. 28 (1932), pp. 45–52, and vol. 29 (1933), pp. 88–94.

§§ J. A. Todd, ibid., vol. 29 (1933), pp. 52–68.

‖‖ H. D. Ursell, ibid., vol. 25 (1929), pp. 31–38.

¶¶ H. W. Richmond, Journal of the London Mathematical Society, vol. 7 (1932), pp. 113–117.

*** E. K. Wakeford, Proceedings of the London Mathematical Society, (2), vol. 21 (1933), pp. 89–113.

††† J. Eiesland, Palermo Rendiconti, vol. 54 (1930), pp. 335–365.

single $[n-2]$ of the base primes. He gives much more information concerning questions discussed by Wong.* Particular attention is given to $n=5$. Various particular cases are included. Another derivation of the equation of V_{n-1}^{n-1} is given by Rupp.†

9. *Transformations Defined by Projective Systems.* An important type of Cremona transformations, applicable to spaces of any number of dimensions and including a large number of cases in each, can be illustrated by the simplest case as follows.

Given four points A, B, C, D in the plane, and a line d in the plane, not passing through any of them. Make the conics of the pencil projective with the points of d. A point y fixes a conic and a point Z on d. The line yZ meets the conic belonging to Z in a point y'. The correspondence y, y' is an involutorial Cremona transformation of the type $c_1 \sim c_7 : 4^3 3^2$ in which the three double points are all on d. They are defined as those points Z which lie on the conics associated with them. When one of these conics is composite, the component through Z divides out, lowering the order of the transformation. In the plane, all of these are particular cases of the Geiser involution. Interesting variations occur when the four points approach each other in various ways. The projectivity may be replaced by a $(1, k)$ correspondence, k conics being associated with the same point Z.

In space, the simplest case is that of a pencil of quadrics, the base curve not meeting the line d. Among the possible types are those discussed by Montesano,‡ furnishing an illustration of a congruence of lines being transformed into itself. The pencil of quadrics may be any of the possible types, and the correspondence may be $(1, 1)$ or $(1, k)$. The interesting features are the appearance and behavior of parasitic lines.

The quadrics may be replaced by a pencil of surfaces of order n having d to multiplicity $n-2$. When the residual curve is nonreducible, these cases have all been considered by Carroll.§ The application to pencils having a composite residual base curve has thus far been limited to pencils of cubics having d as a simple base line.

* B. C. Wong, this Bulletin, vol. 34 (1928), pp. 829–832.

† C. A. Rupp, this Bulletin, vol. 35 (1929), pp. 319–320.

‡ D. Montesano, Giornale di Matematiche, vol. 31 (1893), pp. 36–50.

§ E. T. Carroll, American Journal of Mathematics, vol. 54 (1932), pp. 707–717.

The configuration of the homaloids along the various base lines includes an unexpectedly large number of variations. (Rusk.*) It would be well worth while to make the corresponding study of pencils of surfaces of larger order, and to extend it to pencils of primals in hyperspace.

The straight line d may be replaced by a rational curve. Here the possible number of types is limited, but new arrangements of fundamental elements appear. The case of an irreducible residual base curve has been completed, and that of composite curves is limited to the features characteristic of the composite case. (Black.†) Various other types have been considered by Davis,‡ in connection with Cremona transformations contained in a special or non-special linear line complex.

Isolated cases had been discussed earlier by Pieri and by Montesano. The latter proposed to classify involutions according to the product of the order of the complex to which each belongs and the number of pairs of conjugate points on each line. The involution belonging to the general cubic primal of S_4 belongs to a special linear complex, and each line has two pairs; it is probably irrational, but no case has as yet been proved definitely irrational. Illustrations defined by pairs of planes in a curiously simple form were given by Snyder.§ All of these cases are rational. (Dye.‖) Since every line of a surface of a pencil that passes through its associated point is a fundamental line of the second kind, a particularly instructive case is that in which every surface of the pencil is ruled. Through every point of the rational curve d pass a number of generators, hence the surface generated by them is a component of every homaloid. The base curves of the pencil and isolated generators form the entire system of fundamental curves. Along the directrix curves the surfaces have contact of maximum order, furnishing a prob-

* Evelyn T. Carroll ≡ (Mrs.) E. C. Rusk, American Journal of Mathematics, vol. 56 (1934), pp. 96–108.

† A. H. Black, Transactions of this Society, vol. 34 (1932), pp. 795–810; this Bulletin, vol. 40 (1934), pp. 417–420.

‡ H. A. Davis, American Journal of Mathematics, vol. 52 (1930), pp. 58–71, and vol. 53 (1931), p. 72; H. A. Davis and Tyr. Davis, Tôhoku Mathematical Journal, vol. 33 (1931), pp. 53–58.

§ V. Snyder, this Bulletin, vol. 36 (1930), pp. 89–93; Atti, Bologna Congress, vol. 4 (1930) [1928], pp. 13–21.

‖ L. A. Dye, Transactions of this Society, vol. 32 (1930), pp. 251–261.

lem in postulation and equivalence not accounted for by known formulas. Apart from a few isolated cases, all of these types have for directrix system a line and a rational curve of order m meeting it in $m-1$ points. (Snyder.*)

A non-linear null system defined by a cubic surface gives rise to a Cremona involution by associating each surface of a pencil of cubics with the points of a line, each surface passing through its associated point but not containing the line. (Snyder and Schoonmaker.†) Applications of the method to obtain certain space generalizations of the Geiser and Bertini plane involutions have been made by Dye and Sharpe.‡

10. *Double Fives.* B. Segre§ considers the problem: Is it possible to find a pyramid (simplex) which is simultaneously inscribed in and circumscribed to a quadric primal Q in S_4? Let the faces α_i touch Q in B_i, and four faces meet in A_i. If such a simplex is possible, then both pyramids $A_1 \cdots A_5$ and $B_1 \cdots B_5$ are both inscribed and circumscribed, and are polar conjugates as to Q. It is shown that pyramids do exist. A face (three space) α_5 cuts Q in a point cone, vertex at B_5, and containing the lines B_5A_i, $(i=1, 2, 3, 4)$. On the cone these four generators are equianharmonic. Two pairs of conjugate pyramids are projectively equivalent. One pair is transformed into the other by 120 collineations and by 120 dualities. Together these form a group under which Q remains invariant.

Five associated planes of S_4 (as defined by C. Segre‖) always admit a pair of conjugate pyramids, and consequently define the quadric Q. The five planes are the intersections of the five pairs of opposite faces. By polarity as to Q these are changed into five associated lines, related to the given planes. These lines are the joins of associated vertices of the two conjugate pyramids. If Q be projected stereographically from a point L upon it into an S_3 not passing through L, a fundamental conic γ results. The twenty lines that are projections of A_iB_k, $(i \neq k)$, are all unisecants of γ.

* V. Snyder, Transactions of this Society, vol. 35 (1933), pp. 341–347.

† V. Snyder and H. E. Schoonmaker, American Journal of Mathematics, vol. 54 (1932), pp. 299–304.

‡ L. A. Dye and F. R. Sharpe, Transactions of this Society, vol. 36 (1934), pp. 292–305.

§ B. Segre, Lincei Memorie, (6), vol. 2 (1928), pp. 204–209.

‖ C. Segre, Torino Memorie, (2), vol. 39 (1903), pp. 1–48.

The projections of the points A_i, B_i form a double five of points related to the conic. If γ is projected from any one of them, the resulting quadric cone contains four other points of the double five, and on the cone the generators containing them are always equianharmonic. An irreducible conic has ∞^{10} related double fives of points.

11. *Sets of Five Lines in S_3.* If now the well known mapping of the lines of S_3 on the points of an irreducible quadric primal of S_5 be employed, corresponding to double fives of points relative to the various hyperplane sections of Q, there exist configurations of lines in S_3 which have the following properties.

Let a_i, b_i be two sets of five lines each in S_3 such that (1) they all belong to the same non-special linear complex L, (2) that each line a_i is incident to four lines b_k, but not to b_i, and each line b_i is incident with all lines a_k except a_i. Such a configuration of lines is called a double five. The four points in which a_i meet b_k are equianharmonic.

Given a line b_5 in a non-special linear complex L, and four lines $a_1, \cdots, a_4$ of L meeting it, the reguli defined by these lines a_i, three at a time, contain one and only one line of L, apart from b_5. The necessary and sufficient condition that the four lines thus obtained have only one common transversal is that the set of lines a_i be equianharmonic.

A number of interesting properties follow from this simple construction of a double five of lines in S_3. This result was obtained in terms of lines of S_3 by B. Segre in an earlier paper.* The same result was obtained by Weitzenböck.† (See also Schaake.‡) The general properties of double fours and of double fives were given by Maurer.§

12. *Sets of Oriented Circles in the Plane.* Another interpretation can be given in terms of oriented circles in the plane, mapping each on a point of a general quadric Q primal of S_4, in such manner that those oriented circles which touch each other in the same point are mapped on the points of a line of Q. In a plane are ∞^{10} double fives of oriented circles, such that each circle is

* B. Segre, Rendiconti dei Lincei, (6), vol. 2 (1925), pp. 539–542.

† R. Weitzenböck, Amsterdam Proceedings, vol. 31 (1928), pp. 133–137.

‡ G. Schaake, Amsterdam Proceedings, vol. 31 (1928), pp. 715–717.

§ A. Maurer, Bonn disseration, 1929.

touched by 4 circles, belonging to the quintuple to which the given circle does not belong. Upon the latter circle the four points of contact are equianharmonic. This last property was found by a different method by Barbilian,* and further discussed by Tzitzeica.† The relations between the two interpretations were shown by Weiss.‡ Part of these relations were obtained by Study,§ who also obtains a double five, but these are not equivalent to those here considered, as the complex containing the ten lines is so particularized that the S_4 represented by it in S_5 cuts Q in a quadric Q' having the property that all ten lines touch it.

Gambier‖ applies the ideas of orientation to circles and spheres in contact, including a metric generalization of the Morley-Peterson property to S_n; he¶ makes an application to orthogonal systems by methods of pure geometry, the case $n=4$ producing the results obtained by Barbilian. E. Ciani** points out that the whole idea of double fives is due to him.††

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* D. Barbilian, Bucharest Akademie Bulletin, Mathematics and Physics, vol. 1 (1929), pp. 12–16.

† G. Tzitzeica, Bucharest Akademie Bulletin, Mathematics and Physics, vol. 1 (1929), pp. 16–21.

‡ E. A. Weiss, Amsterdam Proceedings, vol. 35 (1932), pp. 969–978.

§ E. Study, Berlin Sitzungsberichte, 1926, pp. 360–380.

‖ B. Gambier, Comptes Rendus, vol. 190 (1930), pp. 157–159.

¶ B. Gambier, Bulletin des Sciences Mathématiques, vol. 55 (1931), pp. 75–96; Journal de Mathématiques, (9), vol. 11 (1931), pp. 377–387.

** E. Ciani, Annali di Matematica, (3), vol. 8 (1902), pp. 1–21.

†† E. Ciani, Rendiconti dei Lincei, (6), vol. 17 (1933), pp. 215–217.

NOTE ON THE ITERATION OF FUNCTIONS OF ONE VARIABLE*

BY MORGAN WARD

1. *Introduction.* Let $E(x)$ be a real-valued function of the real variable x for some specified range, and let

$$E_0(x) = x,\ E_1(x) = E(x),\ \cdots,\ E_{n+1}(x) = E(E_n(x)),\ \cdots$$

represent its successive iterations. The interpolation problem of defining $E_n(x)$ for non-integral values of n was discussed some time ago by A. A. Bennett,† who reduced it formally to the solution of the functional equation

$$\psi(x + 1) = E(\psi(x)). \tag{1}$$

For if $\psi(x)$ satisfies (1) and if n is any positive integer,

$$\psi(x + n) = E_n(\psi(x)). \tag{2}$$

Hence on writing $\psi^{-1}(x)$ for x, where $\psi^{-1}(x)$ denotes an inverse of the function $\psi(x)$, we obtain the formula

$$E_n(x) = \psi(\psi^{-1}(x) + n), \tag{3}$$

defining $E_n(x)$ for a continuous range of values of n.

In this note, I propose to give an entirely elementary explicit solution to this problem of interpolation for all functions $E(x)$ subject to the following three conditions:‡

(a). *$E(x)$ is a real, continuous, single-valued function of the real variable x in the range $a \leqq x < \infty$.*

(b). *$E(x) > x$ for all $x \geqq a$.*

(c). *$E(x') > E(x)$ if $x' > x \geqq a$.*

We may remark that the functional equation (1) is merely another form of a famous equation studied by Abel,§

* Presented to the Society, June 20, 1934.

† In two papers in the Annals of Mathematics, (2), vol. 17 (1915–16), pp. 74–75 and pp. 23–60. This second paper contains references to the earlier literature. A. Korkine (Bulletin des Sciences Mathématiques, (2), vol. 6 (1882), pp. 228–242) seems to have been the first to consider this problem.

‡ These conditions are all satisfied by $E(x) = e^x$, the particular case discussed by Bennett in the first paper cited.

§ Works, vol. II, *Posthumous Papers*, 1881, pp. 36–39.

$$\phi(x) + 1 = \phi(f(x)), \tag{4}$$

as Abel himself showed.* Here $f(x)$ is a given function, and $\phi(x)$ is to be determined. This equation has been extensively investigated of late by modern function-theoretic methods.†

2. *A Simplification.* As a preliminary simplification, we may assume that the constant a in condition (a) is zero, and that $E(0)=1$. For if $E(a)\neq 0$, the function $E'(x)=E^2(x+a)/E^2(a)$ satisfies conditions (a), (b), (c) with $a=0$, while $E'(0)=1$, and $E(x)=\pm E(a)(E'(x-a))^{1/2}$. On the other hand, if $E(a)=0$, then $E_2(a)=E(0)>0$ by (b). Hence $E''(x)=E_2(x+a)/E_2(a)$ will satisfy (a), (b), (c) with $a=0$, $E''(0)=1$. Since $E(x)$ is continuous and monotonic increasing, it has a unique inverse $E_{-1}(x)$. Thus, if $E''(x)$ is given, $E(x)=E_{-1}(E_2(a)E''(x-a))$.

From (b) and (c), it follows that for any positive integer n, $E_n(x')>E_n(x)$ if $x'>x$. Since $E_n(x)$ is furthermore continuous by (a), it has a unique inverse which we shall denote by $E_{-n}(x)$. If we write $y=E_n(x)$, then by (b), $y\geqq E_n(0)$, so that $E_{-n}(x)$ is defined only for $x\geqq E_n(0)$. It is easily verified, however, that for any $x\geqq 0$,

$$E_n(E_m(x)) = E_{n+m}(x) \tag{5}$$

for all integral values of n and m, positive or negative, for which the functions are defined.

3. *Solution of* (1). We shall next give a solution of the functional equation (1). Let $[x]$ denote as usual the greatest integer in x so that

$$0 = E_0(0) \leqq x - [x] < E_1(0) = 1. \tag{6}$$

Then

$$\psi(x) = E_{[x]}(x - [x])$$

is a monotonic increasing continuous solution of (1). For

$$\begin{aligned}\psi(x+1) &= E_{[x+1]}(x+1-[x+1]) = E_{[x]+1}(x-[x]) \\ &= E(E_{[x]}(x-[x])) = E(\psi(x)),\end{aligned}$$

* Write (1) in the form $x+1=\psi^{-1}(E(\psi(x)))$. Then on substituting $\psi^{-1}(x)$ for x, we obtain $\psi^{-1}(x)+1=\psi^{-1}(E(x))$.

† See, for example, Picard, *Leçons sur Quelques Equations Fonctionnelles*, 1928, Chapter 4. For more recent papers, see the Zentralblatt für Mathematik under the index *Funktionentheorie: Iterationen.*

and if $x' \geqq x+1$,

$$\psi(x') = E_{[x']}(x' - [x']) \geqq E_{[x']}(0)$$
$$= E_{[x']-1}(1) \geqq E_{[x]}(1) > E_{[x]}(x - [x]) = \psi(x),$$

while if $x+1 > x' > x$,

$$\psi(x') = E_{[x']}(x' - [x']) = E_{[x]}(x' - [x])$$
$$> E_{[x]}(x - [x]) = \psi(x).$$

The continuity of $\psi(x)$ is obvious if x is not an integer n. Also if $x=n$, $\epsilon>0$, it is clear that $\lim_{\epsilon\to 0}\psi(n+\epsilon)=\psi(n)$. On setting $x=n-\epsilon$, $\epsilon>0$, we have $\lim_{\epsilon\to 0}\psi(n-\epsilon)=\lim_{\epsilon\to 0}E_{n-1}(1-\epsilon)$ $=E_{n-1}(1)=E_n(0)=\psi(n)$.

It follows that $\psi(x)$ has a unique inverse $\psi^{-1}(x)$. To determine it, let x be given, and let the positive integer k be determined by the inequality

$$E_k(0) \leqq x < E_k(1). \tag{7}$$

Then

$$\psi^{-1}(x) = E_{-k}(x) + k.$$

For first of all, $\psi^{-1}(x)$ is defined and continuous for all $x \geqq 0$. Secondly, from (7), $0 \leqq E_{-k}(x) < 1$ so that $k = [\psi^{-1}(x)]$, the greatest integer in $\psi^{-1}(x)$. Therefore

$$\psi(\psi^{-1}(x)) = E_k(\psi^{-1}(x) - k) = E_k(E_{-k}(x)) = E_0(x) = x.$$

Thirdly, since $E_{[x]}(0) \leqq \psi(x) < E_{[x]}(1)$,

$$\psi^{-1}(\psi(x)) = E_{-[x]}(\psi(x) + [x]) + [x]$$
$$= E_{-[x]}(E_{[x]}(x - [x])) + [x] = x.$$

We obtain then, on substituting in (3), the final result of this note:

$$E_n(x) = E_{[n+k+E_{-k}(x)]}(n + k + E_{-k}(x) - [n + k + E_{-k}(x)])$$
$$= E_{[n+k+E_{-k}(x)]}(n + E_{-k}(x) - [n + E_{-k}(x)]). \tag{8}$$

Here the integer k is determined by the inequality (7) and the formula is valid for all real values of $n \geqq 0$. The equation (5) may now be shown to hold for non-integral values of m and n.

CALIFORNIA INSTITUTE OF TECHNOLOGY

A NEW PROOF OF A THEOREM OF DENJOY, YOUNG, AND SAKS*

BY E. H. HANSON

In 1915 Denjoy† published the following theorem.

THEOREM. *If $f(x)$ is any continuous function, then, at every point x with the possible exception of a set of measure zero, either* (1) *$f(x)$ has a unique finite derivative; or* (2) *the upper (Dini) derivative on one side is $+\infty$, the lower derivative on the other side is $-\infty$, and the other two extreme derivatives are finite and equal; or* (3) *the two upper derivatives are $+\infty$ and the two lower derivatives are $-\infty$.*

This result was extended to measurable functions by G. C. Young and to unrestricted functions by S. Saks.‡ The basis for the latter proof is the corresponding theorem for monotone functions, that is, every monotone function has a unique finite derivative almost everywhere.§ Another proof, as yet unpublished, is due to H. Blumberg, who uses a general principle for passing from measurable functions to general functions.

The present paper gives a new proof of this theorem, for the general case, in a manner which lays bare the essential simplicity of the necessary reasoning involved. This proof, indeed, is a direct remodeling, by means of a few well known devices that may now be regarded as standardized in analysis, of the intuitive perception of the theorem which readily suggests itself. It appears, too, that there is no real gain in arguing with a special hypothesis such as continuity or measurability.

* Presented to the Society, December 2, 1933. The author wishes to express his appreciation to H. Blumberg for helpful criticism.

† Journal de Mathématiques, (7), vol. 1, pp. 105–240.

‡ G. C. Young, Proceedings of the London Mathematical Society, (2), vol. 15 (1916), pp. 360–384; S. Saks, Fundamenta Mathematicae, vol. 5 (1924), pp. 98–104. See also S. Banach, Comptes Rendus, vol. 173 (1921), pp. 457–459; A. S. Besicovitch, Akademiia Nauk, (6), vol. 19 (1925), pp. 97–122 and 527–540; J. C. Burkill and U. S. Haslam-Jones, Proceedings of the London Mathematical Society, (2), vol. 32 (1931), pp. 346–355, and Quarterly Journal of Mathematics, (Oxford), vol. 4 (1933), pp. 233–239.

§ Due to H. Lebesgue, *Leçons sur l'Intégration*, 1904, p. 128.

It will be sufficient to prove that *an arbitrary function* $f(x)$ *has almost everywhere either* $D^+f(x)$ *and* $D_-f(x)$ *finite and equal, or* $D^+f(x)=+\infty$, $D_-f(x)=-\infty$. For, since $D^-f(x)=-D_-[-f(x)]$ and $D_+f(x)=-D^+[-f(x)]$, a similar statement must hold for $D^-f(x)$ and $D_+f(x)$ and the theorem follows.

We shall consider first the set of points at which D^+ and D_- are not both finite. We begin by showing that the set where $D^+=+\infty$ and $D_-\neq-\infty$ is of measure zero. Denote this set by E. Classifying, we have

$$E=\sum_r E_r,$$

where E_r is the set of points of E at which $D_->r$, and the summation is over all rational values of r. Furthermore

$$E_r=\sum_{n=1}^{\infty} E_r^{(n)},$$

where $E_r^{(n)}$ is the set of points x of E_r at which $f(x)-f(x')>r(x-x')$ for all x' between $x-1/n$ and x. It will suffice to show that, for every pair (r, n), $E_r^{(n)}$ is of measure zero. Let (a, b) be any interval of length less than $1/n$ with right end point b in $E_r^{(n)}$ and denote by S the set of points of $E_r^{(n)}$ interior to (a, b). Let k be any number whatsoever and consider the set T of all intervals (x, x') interior to (a, b) for which the left end point x is in S and $f(x')-f(x)>k(x'-x)$. Every point x of S is the left end point of an arbitrarily small interval of T. Hence, according to the Vitali Covering Theorem,* there exist, for every $\epsilon>0$, a finite number of non-overlapping intervals of T, say $I_\nu=(x_\nu, x_\nu')$, $(\nu=1, 2, \cdots, p)$, such that

$$\left| m_e(S)-\sum_{\nu=1}^{p} m(I_\nu)\right|<\epsilon,$$

where m_e stands for exterior measure and m for measure. Furthermore, if (α, β) is one of the intervals complementary to the I_ν with respect to (a, b), β is a point of $E_r^{(n)}$ and $\beta-\alpha<1/n$. Hence $f(\beta)-f(\alpha)>r(\beta-\alpha)$. Therefore

$$f(b)-f(a)>k\sum_{\nu=1}^{p} m(I_\nu)+r\left[b-a-\sum_{\nu=1}^{p} m(I_\nu)\right].$$

* Use is here made of a slight modification of the form in which the Vitali Theorem is customarily stated in that the points covered are allowed to be end points of the covering intervals.

Substituting for $\sum_{r=1}^{p} m(I_r)$ its equivalent in terms of $m_e(S)$, we obtain

$$f(b) - f(a) > k[m_e(S) + \epsilon'] + r[b - a - m_e(S) - \epsilon'],$$

where $|\epsilon'| < \epsilon$. It follows that

$$f(b) - f(a) \geqq km_e(S) + r[b - a - m_e(S)].$$

Since this inequality holds for arbitrarily large k, we must have $m_e(S) = 0$ and hence $m(E_r^{(n)}) = 0$.

We next note that the set of points at which $D_- = +\infty$ is of measure zero. For, by a theorem of G. C. Young,* except in a denumerable set the upper derivative on either side is greater than or equal to the lower derivative on the other. Hence, almost everywhere in the set of points where $D_- = +\infty$, we have also $D^+ = +\infty$. But the set of points at which $D^+ = +\infty$ and $D_- \neq -\infty$ is of measure zero.

Since $D^+f(x) = -D_-f(-x)$, the measure of the set of points at which either $D^+ = -\infty$ or simultaneously $D_- = -\infty$ and $D^+ \neq +\infty$ is also of measure zero. Summarizing, we can now say that, in the set of points at which either D^+ or D_- is either $+\infty$ or $-\infty$, we necessarily have $D^+ = +\infty$ and $D_- = -\infty$ at every point with the possible exception of a set of measure zero.

There remains to be considered the set of points at which both D^+ and D_- are finite. We shall show that, almost everywhere in this set, $D^+ = D_-$. By the previously mentioned theorem of G. C. Young, $D^+ \geqq D_-$ except in a denumerable set. Accordingly, we now let E denote the set of points at which $D^+ > D_-$, both being finite. Classifying, we have

$$E = \sum_{r_1, r_2, r_3, r_4} E_{r_1, r_2, r_3, r_4},$$

* It is not necessary to refer to this theorem if we wish to use the Vitali Theorem again to show that the set of points at which, for example, $D_- > D^+$ is of measure zero. However, the denumerability of this set can be seen in an elementary way as follows. Denote the set of points at which $D_- > D^+$ by E. Then $E = \sum_{n,r} E_r^{(n)}$, where $E_r^{(n)}$ is the set of points x of E such that for every x' between $x - 1/n$ and x and for every x'' between x and $x + 1/n$, we have

$$\frac{f(x) - f(x')}{x - x'} > r > \frac{f(x'') - f(x)}{x'' - x},$$

and where the summation is over all pairs (n, r), n being a positive integer and r being rational. Since two points of $E_r^{(n)}$ are at least at distance $1/n$ apart, $E_r^{(n)}$, and therefore E, is denumerable.

where E_{r_1, r_2, r_3, r_4} denotes the set of points where $r_1 > D^+ > r_2 > r_3 > D_- > r_4$ and the summation is over all quadruples of rational numbers r_1, r_2, r_3, r_4 such that $r_1 > r_2 > r_3 > r_4$. Moreover,

$$E_{r_1, r_2, r_3, r_4} = \sum_{n=1}^{\infty} E^{(n)}_{r_1, r_2, r_3, r_4},$$

where $E^{(n)}_{r_1, r_2, r_3, r_4}$ denotes the set of points x of E_{r_1, r_2, r_3, r_4} such that, for every x' between $x - 1/n$ and x, we have $f(x) - f(x') > r_4(x - x')$ and, for every x' between x and $x + 1/n$, we have $f(x') - f(x) < r_1(x' - x)$. It will suffice to show that $m(E^{(n)}_{r_1, r_2, r_3, r_4}) = 0$. Let (a, b) be any interval of length less than $1/n$ with end points in $E^{(n)}_{r_1, r_2, r_3, r_4}$ and let S be the set of points of $E^{(n)}_{r_1, r_2, r_3, r_4}$ interior to (a, b). We may substitute r_2 and r_4, respectively, for k and r in the inequality obtained above, and thus have

$$f(b) - f(a) \geqq r_2 m_e(S) + r_4[b - a - m_e(S)].$$

A similar reasoning gives

$$f(b) - f(a) \leqq r_3 m_e(S) + r_1[b - a - m_e(S)],$$

and it results that

$$\frac{m_e(S)}{b - a} \leqq \frac{r_1 - r_4}{r_1 - r_4 + r_2 - r_3}.$$

The right hand member of this inequality being a constant less than 1, it follows that $E^{(n)}_{r_1, r_2, r_3, r_4}$ can have exterior metric density 1 at no point. Hence $m(E^{(n)}_{r_1, r_2, r_3, r_4}) = 0$. This completes the proof of the theorem.

Although in the proof we have tacitly assumed $f(x)$ to be defined on an interval, the method of proof is independent of this assumption and $f(x)$ may be regarded as defined on an arbitrary point set.

JACKSON, OHIO

NOTE ON THE USE OF FRACTIONAL INTEGRATION OF BESSEL FUNCTIONS*

BY H. P. THIELMAN

1. *Introduction.* In a recent paper, W. O. Pennell† pointed out that under certain conditions fractional integration and differentiation of trigonometric functions will lead to Bessel functions and vice versa. Making use of this fact he was able to convert known expansions in sines and cosines into expansions in Bessel functions, and known expansions in Bessel functions into trigonometric expansions by the simple process of term-by-term differentiation and integration. In this note it is shown that in some cases fractional integration of Bessel functions leads to the squares of such functions. Use is then made of this fact in order to derive a number of expansions in squares of Bessel functions by fractionally integrating known expansions in Bessel functions.

2. *Fractional Integration of Bessel Functions.* Fractional integration‡ with the Heaviside operator p is defined by the equation

$$p^{-\nu}f(x) = \int_0^x \frac{(x-t)^{\nu-1}}{\Gamma(\nu)} f(t)dt, \tag{1}$$

where $\nu > 0$. Fractional differentiation is given by

$$p^{\nu}f(x) = \frac{d^b}{dx^b}\int_0^x \frac{(x-t)^{c-1}}{\Gamma(c)} f(t)dt, \tag{2}$$

where $\nu > 0$, $0 < c < 1$, b is a positive integer, and $\nu = b - c$.

If in the well known Neumann's integral§

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* Presented to the Society, April 7, 1934.

† W. O. Pennell, *The use of fractional integration and differentiation for obtaining certain expansions in terms of Bessel functions or of sines and cosines*, this Bulletin, vol. 38 (1932), pp. 115–122.

‡ For a bibliography on fractional integration and differentiation see H. T. Davis, *The application of fractional operators to functional equations*, American Journal of Mathematics, vol. 49 (1927), pp. 123–142.

§ See G. N. Watson, *Theory of Bessel Functions*, p. 150.

$$J_\mu(z)J_\nu(z) = \frac{2}{\pi}\int_0^{\pi/2} J_{\mu+\nu}(2z\cos\theta)\cos(\mu-\nu)\theta d\theta,$$

where $\mu+\nu>-1$, we set $\mu=\nu$, $z=ax^{1/2}$, $\cos\theta=t^{1/2}x^{1/2}$, we obtain

$$J_\nu^2(ax^{1/2}) = \frac{1}{\pi}\int_0^x J_{2\nu}(2at^{1/2})(x-t)^{-1/2}t^{-1/2}dt, \qquad (\nu>-1/2),$$

which by (1) is seen to be equivalent to

$$(3) \qquad p^{-1/2}x^{-1/2}J_{2\nu}(2ax^{1/2}) = \pi^{1/2}J_\nu^2(ax^{1/2}), \qquad (\nu>-1/2).$$

Here $J_\nu(z)$ is the Bessel function of the first kind of order ν.

By transposing the operator in (3), we get

$$(4) \qquad p^{1/2}J_\nu^2(ax^{1/2}) = \pi^{-1/2}x^{-1/2}J_{2\nu}(2ax^{1/2}), \qquad (\nu>-1/2).$$

3. *Fractional Integration of Expansions in Bessel Functions.* The *Fourier-Bessel* series for $f(x)$ is given by

$$(5) \qquad f(x) = \sum_{n=1}^{\infty} a_n J_\nu(j_n t)dt, \qquad (0<x<1, \nu>-1/2),$$

where

$$(6) \qquad a_n = \frac{2}{J_{\nu+1}^2(j_n)}\int_0^1 tf(t)J_\nu(j_n t)dt,$$

and where j_n is the nth positive root of $J_\nu(x)=0$. From (5) it follows that

$$(7) \qquad x^{-1/2}f(x^{1/2}) = \sum_{n=1}^{\infty} a_n x^{-1/2}J_\nu(j_n x^{1/2}).$$

The conditions on $f(x)$ under which the series (7) is uniformly convergent, or under which at least the sum of n terms of this series remains bounded for all n and all x in the interval $0\leqq x\leqq 1$, are clearly stated in Lemma 1 and Theorem I of a paper by C. N. Moore.* Assuming that $f(x)$ satisfies these conditions, we can integrate the series (7) fractionally† term by term. We get, applying equation (3),

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* C. N. Moore, *On the uniform convergence of the developments in Bessel functions*, Transactions of this Society, vol. 12 (1911), pp. 181–206.

† W. O. Pennell, loc. cit.

$$p^{-1/2}x^{-1/2}f(x^{1/2}) = \pi^{1/2}\sum_{n=1}^{\infty} a_n J_{\nu/2}^2\left(\frac{j_n x^{1/2}}{2}\right).$$

If we put

$$\phi(x^{1/2}) = p^{-1/2}x^{-1/2}f(x^{1/2}), \tag{8}$$

then

$$f(x^{1/2}) = x^{1/2}p^{1/2}\phi(x^{1/2}),$$

and

$$\phi(x) = \pi^{1/2}\sum_{n=1}^{\infty} a_n J_{\nu/2}^2\left(\frac{j_n x}{2}\right), \qquad (0 < x < 1,\ \nu \geqq 1), \tag{I}$$

where a_n is given by (6).

If we apply the preceding reasoning to the Dini series*

$$f(x) = \sum_{n=1}^{\infty} b_n J_\nu(\lambda_n x),$$

where λ_n is the nth positive root of

$$z^{-\nu}\{zJ_\nu'(z) + HJ_\nu(z)\} = 0, \qquad (\nu > -1/2),$$

H any constant, and

$$b_n = \frac{2\lambda_n^2 \int_0^1 tf(t)J_\nu(\lambda_n t)dt}{(\lambda_n^2 - \nu^2)J_\nu^2(\lambda_n) - \lambda_n^2 J_\nu'^2(\lambda_n)},$$

we obtain the expansion

$$\phi(x) = \pi^{1/2}\sum_{n=1}^{\infty} b_n J_{\nu/2}^2\left(\frac{\lambda_n x}{2}\right), \qquad (0 < x < 1,\ \nu \geqq 1). \tag{II}$$

The function $\phi(x)$ is such that if $f(x^{1/2}) = x^{1/2}p^{1/2}\phi(x^{1/2})$, then $f(x)$ satisfies the conditions sufficient for a function to be expansible in a Dini series which may be integrated term by term† in the interval $0 \leqq x \leqq 1$.

Let $f(x)$ be representable by a Schlömilch series of the type‡

$$f(x) = \pi^{1/2}2^{\nu-1}\sum_{n=1}^{\infty} \frac{a_n}{n^{\nu-1}} J_\nu(nx), \tag{9}$$

* See Watson, loc. cit., pp. 597–605.

† C. N. Moore, loc. cit., Lemma 1, Theorem I.

‡ Pennell, loc. cit., p. 120.

which is assumed to be such that it may be integrated term-by-term in the interval $0 \leqq x \leqq \pi$. Making use of the method by which (I) was obtained from (5), we find the expansion

$$\text{(III)} \qquad \phi(x) = \pi 2^{\nu-1} \sum_{n=1}^{\infty} \frac{a_n}{n^{\nu-1}} J_{\nu/2}^2\left(\frac{nx}{2}\right), \qquad (0 < x < \pi,\ \nu \geqq 1),$$

where

$$\phi(x^{1/2}) = p^{-1/2} x^{-1/2} f(x^{1/2}).$$

If the above method is applied to Neumann and Kapteyn series, well known expansions in terms of squares of Bessel functions are obtained. Expansions (I), (II), and (III) have seemingly never been published.

THE OHIO STATE UNIVERSITY

NOTE CONCERNING GROUP POSTULATES*

BY RAYMOND GARVER

Let there be given a set of elements $G(a, b, c, \cdots)$ and a rule of combination, which may be called multiplication, by which any two elements, whether they be the same or different, taken in a specified order, determine a unique result which may or may not be an element of G. This system is called a group if it satisfies certain postulates; various sets of postulates have been given by different writers, and such matters as the independence of postulates and relations between sets of postulates have been pretty thoroughly covered. Most of this work was done in this country in the early part of the present century.†

It seems, however, that one interesting and rather important

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† See Pierpont, Annals of Mathematics, (2), vol. 2 (1900), p. 47; Moore, Transactions of this Society, vol. 3 (1902), pp. 485–492, vol. 5 (1904), p. 549 and vol. 6 (1905), pp. 179–180; Huntington, this Bulletin, vol. 8 (1902), pp. 296–300 and 388–91 and Transactions of this Society, vol. 4 (1903), p. 30, vol. 6 (1905), pp. 34–35 and 181–197; Dickson, Transactions of this Society, vol. 6 (1905), pp. 198–204.

question has been left unanswered, namely, the question of the independence of the following set of postulates:

I. *If a and b* are elements of G, the product ab is an element of G.*

II. *If a, b, c, ab, bc, $(ab)c$, $a(bc)$ are all elements of G, then $(ab)c=a(bc)$.*†

III. *If a and b are elements of G, there exists an element x of G such that $ax=b$.*

IV. *If a and b are elements of G, there exists an element y of G such that $ya=b$.*

This postulate set is a modification of that given by Weber.‡ Weber defined a finite group by I, II, and two further postulates, and then deduced III and IV, and the uniqueness of the elements x and y of III and IV, as theorems for finite groups. Noting that III and IV, with uniqueness, could not be so deduced from his postulates when the number of elements was infinite, he then added them to his set of postulates to define an infinite group. His procedure was perfectly natural, though it led to several redundancies. Huntington, in 1902, actually exhibited these redundancies, though he did not emphasize having done so until 1905.§ Moore, also in 1902, used the postulate system I, II, III, IV just as it is written above, apparently for the first time.‖ He calls it W_1', and states explicitly, on page 489, "For W_1', the independence of the postulates is an open question." And it seems that the question has not as yet been answered.

However, Huntington, in 1902, replaced II by the following stronger postulate:

II′. *If a, b, c, ab, bc, $(ab)c$ are all elements of G, then $(ab)c=a(bc)$.*

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* The symbols a, b, $\cdots$ as used in the postulates need not represent *distinct* elements of G.

† This postulate, the associative law, is sometimes written "If a, b, c are all elements of G, then $(ab)c=a(bc)$." This form is satisfactory in case II is never to be thought of apart from I.

‡ *Lehrbuch der Algebra*, vol. 2, 1896, pp. 3–4.

§ This Bulletin, vol. 8 (1902), pp. 296–300; Transactions of this Society, vol. 6 (1905), p. 182.

‖ Transactions of this Society, vol. 3 (1902), pp. 485–492.

He was then* able to deduce I from II′, III, IV. I propose to show that the use of II′ is not essential, that is, that I can be deduced from II, III and IV. The proof, in fact, is simpler than Huntington's proof that I followed from II′, III, IV. It may be written down as follows:

Assume that a and b are elements of G.

(1) By IV, $\exists\ e$ in G such that $ea=a$.
(2) By IV, $\exists\ a'$ in G such that $a'a=e$.
(3) By III, $\exists\ x$ in G such that $a'x=b$.

We wish to show that $ab=x$.
(4) By III, $\exists\ p$ in G such that $ap=x$.
(5) By III, $\exists\ q$ in G such that $eq=p$.
(6) By III, $\exists\ r$ in G such that $ar=q$.
(7) By (1) and (6), $(ea)r=ar=q$.
(8) By (6) and (5), $e(ar)=eq=p$.
(9) By (7), (8) and II, $p=q$.
(10) By (5) and (9), $ep=p$.
(11) By (2) and (10), $(a'a)p=ep=p$.
(12) By (4) and (3), $a'(ap)=a'x=b$.
(13) By (11), (12) and II, $p=b$.
(14) By (4) and (13), $ab=x$.†

In the proof as I originally wrote it down, I deduced two lemmas from II, III, IV before obtaining the proof of postulate I. It is perhaps worth while noting that they can be obtained directly from II, III, IV.

LEMMA I. *There exists a unique element e, such that, for every element a, $ea=ae=a$.*

LEMMA II. *For every element a there exists a unique element a′ such that $a'a=aa'=e$.*

However, the referee noted that the deduction of these Lemmas in their entirety introduced a few unnecessary steps; his suggestion that the proof be arranged in more direct form is followed above.

We may thus define a group by three postulates, II, III, and

* This Bulletin, vol. 8 (1902), pp. 296–300.

† If the number of elements of G is finite, it is easily seen that I is a direct consequence of either III or IV.

IV. Further, no other set of three postulates chosen from I, II, III, and IV will suffice. For I, II and III, but not IV, or I, II, and IV, but not III, are satisfied by multiple groups, as defined by Clifford, which are not in general groups.* The simplest example of a system satisfying I, II, and III, but not IV, is given by the multiplication table.

	e	a
e	e	a
a	e	a

And II certainly does not follow from I, III, and IV. If the number of elements of G is n, we may give the multiplication table for G by means of a square array, n rows by n columns. If all the entries in this array are elements of G, and if each row, and each column, of the array contains every element of G, I, and III, and IV are satisfied. Yet if n is greater than 2, II need not hold; the simplest example is

	a	b	c
a	a	b	c
b	c	a	b
c	b	c	a

THE UNIVERSITY OF CALIFORNIA AT LOS ANGELES

* Annals of Mathematics, (2), vol. 34 (1933), pp. 865–871.

LINEAR FUNCTIONAL OPERATIONS ON FUNCTIONS HAVING DISCONTINUITIES OF THE FIRST KIND*

BY H. S. KALTENBORN

The most general form of a linear continuous functional operation on a function $f(x)$ continuous on (a, b) has been expressed by F. Riesz† as a Stieltjes integral of $f(x)$ with respect to a function of bounded variation. By virtue of the extension theorem of Hahn,‡ this result can be extended to the case where $f(x)$ has discontinuities of the first kind on (a, b). In establishing this more general case, the modified form of the integral considered by B. Dushnik§ is used instead of the ordinary Stieltjes integral. The expression obtained for $T[f]$ reduces to that given by Riesz when the function $f(x)$ is continuous since in this case the modified integral agrees with the ordinary integral.

The following definitions and properties enter into the statement and proof of the theorem. Let f denote an element of a linear vector space S of norm $\|f\|$;¶ then an operation T on S is linear if

$$T[k_1f_1 + k_2f_2] = k_1T[f_1] + k_2T[f_2], \tag{1}$$

where the k's are any constants; and T is bounded if there exists an $M>0$ depending on T such that

$$|T[f]| \leqq M\|f\| \quad \text{for every } f \text{ of } S. \tag{2}$$

The condition of boundedness (2) is equivalent** to the property of continuity

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* Presented to the Society, April 6, 1934.

† Annales de l'École Normale Supérieure, (3), vol. 31 (1914), pp. 9–14. Two new proofs of this theorem have been given recently by T. H. Hildebrandt and I. J. Schoenberg, Annals of Mathematics, vol. 34 (1933), pp. 317–319.

‡ See S. Banach, *Théorie des Opérations Linéaires*, 1932, pp. 55, 59.

§ Dissertation, University of Michigan, 1931.

¶ See T. H. Hildebrandt, *Linear functional transformations in general spaces*, this Bulletin, vol. 37 (1931), p. 186.

** See S. Banach, loc. cit., p. 54.

$$\lim_{n\to\infty} T[f_n] = T[f], \quad \text{when} \quad \lim_{n\to\infty} \|f_n - f\| = 0. \tag{3}$$

The class C of all continuous functions on (a, b), and the class D of functions having discontinuities of the first kind constitute linear vector spaces whose norms are the maximum and least upper bound, respectively, of $|f(x)|$ on (a, b). The modified form of the integral that will be used is defined by

$$\int_a^b f d\alpha = \lim \sum_{i=1}^{n} f(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})], \tag{4}$$

where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ is any subdivision of (a, b), ξ_i is an arbitrary *inner* point of (x_{i-1}, x_i), and the limit is taken as the subdivision is made successively finer.* The symbol $\int_a^b f d\alpha$ will be used throughout this paper to denote an integral of this type.

An analog of Riesz's theorem for functions of class D is then contained in the following theorem.

THEOREM. *Every linear continuous operation T applied to a function $f(x)$ of D may be expressed in the form*

$$T[f] = \int_a^b f d\psi + \sum_{i=0}^{\infty} [f(c_i) - f(c_i - 0)]\phi(c_i), \tag{5}$$

where $c_0, c_1, c_2, \cdots$ denote the points of discontinuity of $f(x)$, $c_0 = a$, $\psi(x)$ and $\phi(x)$ depend only on T and are such that $\psi(x)$ is of bounded variation and $\phi(x)$ vanishes except at a denumerable set of points and the sum of the absolute values of $\phi(x)$ at these points is finite.

It will be noted that the expression for $T[f]$ depends on subdivisions of (a, b) into sets of open intervals and single points rather than on subdivisions into closed intervals as is the case when $f(x)$ is continuous and the ordinary integral is used. The first term on the right in (5) corresponds to the open intervals of the subdivision, and the second term arises from the points of discontinuity of $f(x)$. If $f(x)$ is continuous, the second term vanishes, and the expression for $T[f]$ reduces to

* See H. L. Smith, Transactions of this Society, vol. 27 (1925), pp. 491–492. The integral (4) differs from Smith's integral $FS\int f d\alpha$ in that ξ_i is restricted to be an inner point of (x_{i-1}, x_i).

the modified integral $\int_a^b f d\psi$. Now the ordinary Stieltjes integral exists since $f(x)$ is continuous and $\psi(x)$ is of bounded variation, and since the modified integral equals the ordinary integral when the latter exists, it follows that the form for $T[f]$ in this case is the same as that obtained by Riesz.

To obtain the functions $\phi(x)$ and $\psi(x)$ of (5), introduce the functions $g(t, x) = 1$ or 0 according as $t = x$ or $t \neq x$, and $h(t, x) = 1$ or 0 according as $a \leqq t \leqq x$ or $x < t \leqq b$, and then place

$$\phi(x) = T[g(t, x)], \qquad \psi(x) = T[h(t, x)]. \tag{6}$$

The properties of $\phi(x)$ and $\psi(x)$ stated in the theorem follow from these definitions. Thus if $x_1, x_2, \cdots, x_n$ is any finite set of points of (a, b), then from (2) and the definition of $g(t, x)$,

$$\begin{aligned}\sum_{i=1}^{n} |\phi(x_i)| &= \sum_{i=1}^{n} \operatorname{sgn} \phi(x_i) T[g(t, x_i)] \\ &\leqq M \left\| \sum_{i=1}^{n} \operatorname{sgn} \phi(x_i) g(t, x_i) \right\| = M.\end{aligned}$$

Consequently $\phi(x)$ vanishes except at a denumerable set of points, and the sum of the absolute values of $\phi(x)$ at these points is $\leqq M$. The function $\psi(x)$ is of bounded variation, since for any subdivision of (a, b),

$$\begin{aligned}\sum_{i=1}^{n} |\psi(x_i) - \psi(x_{i-1})| &= \sum_{i=1}^{n} |T[h(t, x_i) - h(t, x_{i-1})]| \\ &\leqq M \left\| \sum_{i=1}^{n} [h(t, x_i) - h(t, x_{i-1})] \right\| = M.\end{aligned}$$

The given function $f(x)$ may be uniformly approximated on (a, b) by means of a sequence of functions of the type

$$\begin{aligned}f_\sigma(x) &= \sum_{i=1}^{n} f(\xi_i)[h(x, x_i) - h(x, x_{i-1}) - g(x, x_i)] \\ &\quad + \sum_{i=0}^{n} f(x_i) g(x, x_i) \\ &= \sum_{i=1}^{n} f(\xi_i)[h(x, x_i) - h(x, x_{i-1})] \\ &\quad + \sum_{i=0}^{n} [f(x_i) - f(\xi_i)] g(x, x_i),\end{aligned} \tag{7}$$

where $\sigma=(a=x_0<x_1<x_2<\cdots<x_n=b)$ is a finite subdivision of (a, b), ξ_i is an arbitrary inner point of (x_{i-1}, x_i), and $f(\xi_0)=0$. For if $\epsilon>0$ denotes an arbitrarily small quantity, then a subdivision σ_ϵ of (a,b) can be found* so that the oscillation of $f(x)$ on the interior of any subinterval (x_{i-1}, x_i), $(i=1, 2, \cdots, n)$, is less than ϵ. Since $f_\sigma(x_i)=f(x_i)$, and $f_\sigma(x)=f(\xi_i)$ for $x_{i-1}<x<x_i$, $(i=1, 2, \cdots, n)$, then if σ is a finer subdivision than σ_ϵ, $|f(x)-f_\sigma(x)|<\epsilon$ for any x on (a, b), and hence $\lim f_\sigma(x)=f(x)$ uniformly. Therefore, from (3),

$$\lim T[f_\sigma(x)] = T[f(x)].$$

Applying T to (7), and making use of (6), we get

$$T[f_\sigma] = \sum_{i=1}^{n} f(\xi_i)[\psi(x_i) - \psi(x_{i-1})] + \sum_{i=0}^{n} [f(x_i) - f(\xi_i)]\phi(x_i).$$

The limit of the first sum on the right exists† and is equal to $\int_a^b f\,d\psi$. Therefore the limit of the second sum on the right exists, and we have

$$T[f] = \int_a^b f\,d\psi + \lim \sum_{i=0}^{n} [f(x_i) - f(\xi_i)]\phi(x_i). \tag{8}$$

Let Q denote the sum on the right for a subdivision finer than σ_ϵ. Then $|f(x_i-0)-f(\xi_i)|<\epsilon$, $(i=0, 1, 2, \cdots, n)$, and hence

$$\left| Q - \sum_{i=0}^{n} [f(x_i) - f(x_i - 0)]\phi(x_i) \right|$$

$$= \left| \sum_{i=0}^{n} [f(x_i - 0) - f(\xi_i)]\phi(x_i) \right|$$

$$\leqq \epsilon \sum_{i=0}^{n} |\phi(x_i)| \leqq \epsilon M,$$

from which

$$\lim Q = \lim \sum_{i=0}^{n} [f(x_i) - f(x_i - 0)]\phi(x_i).$$

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* H. Hahn, *Theorie der Reellen Funktionen*, vol. 1, 1921, p. 217.

† Dushnik, loc. cit.

Since $|f(c_i)-f(c_i-0)|<\epsilon$ for every c_i which is not a point of σ_ϵ, then there is an integer $m \leqq n$ such that, if $\sum'$ denotes summation over the points of σ which are not points of discontinuity of $f(x)$,

$$\left|\sum_{i=0}^{\infty}[f(c_i)-f(c_i-0)]\phi(c_i)-\sum_{i=0}^{n}[f(x_i)-f(x_i-0)]\phi(x_i)\right|$$

$$=\left|\sum_{i=m}^{\infty}[f(c_i)-f(c_i-0)]\phi(c_i)-\sum'[f(x_i)-f(x_i-0)]\phi(x_i)\right|$$

$$\leqq \sum_{i=m}^{\infty}|f(c_i)-f(c_i-0)||\phi(c_i)|$$

$$+\sum'|f(x_i)-f(x_i-0)||\phi(x_i)| \leqq 2\epsilon M.$$

Therefore,

$$\lim Q=\sum_{i=0}^{\infty}[f(c_i)-f(c_i-0)]\phi(c_i).$$

Substituting this value of lim Q in (8), we obtain (5), and thus the theorem is proved.

If the function $\chi(x)=T[k(t,x)]$, where $k(t,x)=0$ or 1 according as $a \leqq t<x$ or $x \leqq t \leqq b$, is used instead of $\psi(x)$, the expression

$$T[f]=\int_a^b f d\chi+\sum_{i=0}^{\infty}[f(c_i)-f(c_i+0)]\phi(c_i) \tag{9}$$

is obtained instead of (5). To obtain a more general form of $T[f]$, which includes both (5) and (9), multiply (5) by λ and (9) by $1-\lambda$, and add. Then

$$T[f]=\int_a^b f d\theta+\sum_{i=0}^{\infty}[f(c_i)-\{\lambda f(c_i-0)+(1-\lambda)f(c_i+0)\}]\phi(c_i), \tag{10}$$

where $\theta(x)=\lambda\psi(x)+(1-\lambda)\chi(x)$. Certain conditions under which $T[f]$ can be expressed directly as a modified integral without any corrective terms being necessary may be seen at once from the equation (10). This is the case if T is such that $\phi(x)\equiv 0$, or if*

* This case is considered by H. Hahn, Monatshefte für Mathematik und Physik, vol. 32 (1922), p. 53.

$f(x)=\lambda f(x-0)+(1-\lambda)\ f(x+0)$. In particular, $T[f]$ is expressible just as an integral if T operates on the class of functions $f(x)$ which are continuous on the right (or on the left) throughout (a, b), or for which $f(x)=(1/2)(f(x-0)+f(x+0))$ at every point.

If $\phi(x)=(1/2)(\psi(x+0)-\psi(x-0))$ at every point, for the operation T, then $T[f]$ is expressible directly as a Stieltjes mean integral,

$$T[f] = (m)\int_a^b f\,d\alpha, \tag{11}$$

where $\alpha(x)=2\psi(x)-\psi(x+0)$, and the integral is obtained from (4) by replacing $f(\xi_i)$ by $(1/2)(f(x_{i-1})+f(x_i))$. Comparing (5) and (11) for $f(x)=g(t, x)$ and $f(x)=h(t, x)$, we get as necessary and sufficient conditions for their equivalence,

$$\begin{aligned} \phi(x) &= \tfrac{1}{2}[\alpha(x+0)-\alpha(x-0)], \\ \psi(x) &= \tfrac{1}{2}[\alpha(x+0)+\alpha(x)]. \end{aligned} \tag{12}$$

Since $\alpha(x)$ is continuous except at a denumerable set of points, it follows from the second of equations (12) that $\psi(x+0)=\alpha(x+0)$ and $\psi(x-0)=\alpha(x-0)$. By means of these relations, (12) can be reduced to the expressions given above for $\phi(x)$ and $\psi(x)$.

It is sometimes possible to express $T[f]$ in terms of the Young modification of the Stieltjes integral,* which is defined by

$$\begin{aligned} Y\int_a^b f\,d\alpha = \lim \sum_{i=1}^{n} \{ & f(\xi_i)[\alpha(x_i-0)-\alpha(x_{i-1}+0)] \\ & + f(x_{i-1})[\alpha(x_{i-1}+0)-\alpha(x_{i-1})] \\ & + f(x_i)[\alpha(x_i)-\alpha(x_i-0)]\}, \quad (x_{i-1}<\xi_i<x_i). \end{aligned} \tag{13}$$

Conditions for the equivalence of (5) and (13) are found as above by comparing the values of these expressions for the same specializations of the function $f(x)$. They are

* W. H. Young, *Integration with respect to a function of bounded variation*, Proceedings of the London Mathematical Society, (2), vol. 13 (1913), p. 113. As defined by Young, the integral is obtained by taking the limit as the maximum length of the subintervals approaches zero. The integral thus defined is of a weaker type than that given in (13).

$$(14) \qquad \phi(x) = \psi(x+0) - \psi(x-0), \qquad \psi(x) = \psi(x+0).$$

The conditions (14) place a greater restriction on T than the corresponding conditions for expressing T as a single Stieltjes integral of the Dushnik type or of the mean type. Consequently the Young-Stieltjes integral is not as well adapted to representing an operation $T[f]$ as these other modifications. The same remark applies to the Lebesgue-Stieltjes integral, which agrees with the Y-integral in this case.

The fact that a linear functional operation $T[f]$ on the class of functions having discontinuities of the first kind is not always expressible as a Lebesgue-Stieltjes integral or as an ordinary Stieltjes integral of $f(x)$ with respect to a function of bounded variation may be shown in a different manner. Let $f_n(x) = 1$ or 0 according as $0 \leqq x \leqq 1/n$ or $1/n < x \leqq 1$, and let $f_0(0) = 1$, $f_0(x) = 0$ elsewhere in $(0, 1)$; and take as the operation T the mean integral, $(m)\int_0^1 f\,d\alpha$, of f with respect to a function of bounded variation $\alpha(x)$ which is discontinuous on the right at $x = 0$. Then $f_1 \geqq f_2 \geqq f_3 \geqq \cdots \to f_0$, and

$$(m) \qquad \int_0^1 f_0\,d\alpha = \frac{1}{2}\,[\alpha(0+0) - \alpha(0)],$$

$$(m) \qquad \int_0^1 f_n\,d\alpha = \frac{1}{2}\left[\alpha\left(\frac{1}{n}\right) + \alpha\left(\frac{1}{n}+0\right)\right] - \alpha(0)$$

$$\to \alpha(0+0) - \alpha(0),$$

so that $(m)\int f_n\,d\alpha$ does not converge to $(m)\int f_0\,d\alpha$. But the LS-integral and the ordinary Stieltjes integrals possess the property that $\int f_n\,d\alpha$ converges to $\int f_0\,d\alpha$ if $f_n \to f_0$ monotonically,* and therefore these integrals cannot be used to represent this operation T.

THE UNIVERSITY OF MICHIGAN

* See P. J. Daniell, Annals of Mathematics, (2), vol. 19 (1918), p. 280.

MAGIC CIRCLES

BY C. G. P. KUSCHKE

If at every one of the n^2+1 points defined by n concentric circles and n radii is placed one of the first n^2+1 integers such that every circle with its center carries the same sum as every radius, I shall call this figure a system of magic circles of order n. Every magic square of order n gives rise to two such circular systems, for the square with its n rows and n columns may be transformed into concentric circles, the rows becoming the circles and the columns the radii; all that is necessary is to assign to the center C the number n^2+1 or 1; in the latter case all integers of the magic square must be raised by unity. Such systems I shall exclude here and all other systems I shall then call true circular systems.

The circles will be denoted by letters $a, b, c, \cdots$, (a being the smallest circle), and the radii by the integers 1, 2, 3, $\cdots$ so that, for instance, the integer standing at the intersection of circle d with radius 6 will be denoted by d_6.

All systems obtained by an interchange of rows, columns, or row and columns in any particular system will form a group and that system in the groups with $a_1=1$, $a_2<a_3<a_4<\cdots$, $b_1<c_1<d_1<\cdots$, and $b_1<a_2$ will be considered the representative of this group. Every group contains, then, $2(n!)^2(1/n)$ distinct systems and just one representative, because every system can belong to only one group and two systems are considered to be identical, if one may be obtained from the other by rotating merely the paper on which the figure is drawn. The sum of all integers in a system of order n is

$$S_n = \sum_{i=1}^{n^2+1} i = \tfrac{1}{2}(n^2+1)(n^2+2),$$

and the magic sum is

$$s_n = (1/n)\left[S_n + (n-1)M\right] = 1 + b_1 + c_1 + d_1 + \cdots,$$

where M is the element at the center, so that we have now two restrictions on M, namely, $1<M\leqq n^2$, and $M\equiv 1 \pmod{n}$. For each M, all possibilities for the set $(b_1, c_1, d_1, \cdots)$ must be

found and from the remaining integers in each case all chances for the set $(a_2, a_3, a_4, \cdots)$. What is left to be filled out may be stretched into a square of $n-1$ rows and $n-1$ columns, but that square would not be a magic square in the common sense for it has no magic constant; the magic number is a variable in the sense that, while every row and column must show a definite sum, this sum is different for any two rows and any two columns. I shall call such a square a variable magic square. Every such variable magic square of order $n-1$ gives rise to just one circular system of order n, for it must fit in this circular system so that no permutation of rows and columns is possible.

For $n=2$, there is no circular system, for our above conditions would lead to $b_1=a_2$. For $n=3$, we have $M\equiv 1 \pmod 3$; therefore $M=4$ or 7. If $M=4$, $s=21$, $b_1+c_1=16$, so that the pair $(b_1, c_1)=(6, 10)$ or $(7, 9)$. The pair $(b_1, c_1)=(6, 10)$ gives $(a_2, a_3)=(7, 9)$, as $a_2+a_3=16$ too; and $(b_1, c_1)=(7, 9)$ leaves no further room for (a_2, a_3) under our above conditions for a representative. It remains, then, to fill out a variable magic square of order two; the rows must show the sums 11 and 7, and the columns the sums 10 and 8, and to do this we have for our disposal the integers 2, 3, 5, 8. It follows, then, that there is only one chance, namely, $b_2=8$, $b_3=3$, $c_2=2$, $c_3=5$. If $M=7$, $b_1+c_1=15$, so that $(b_1, c_1)=(5, 10)$ and $(a_2, a_3)=(6, 9)$. We have left for the variable magic square of order 2 the integers 2, 8, 3, 4, leading to only one chance, namely, $b_2=8$, $b_3=3$, $c_2=2$, $c_3=4$.

For $n=4$, $M=5$, 9, or 13. If $M=5$, $s_4=42$. There are 17 possibilities for the triad (b_1, c_1, d_1), four of which leave no room for the triad (a_2, a_3, a_4). The number of groups is 125, so that we have $2(4!)^2(1/4)125=36000$ systems. For $M=9$, $s_4=45$. There are 18 possibilities for (b_1, c_1, d_1), four of which give no triad (a_2, a_2, a_4). The number of groups in this case is 517. There are $2(4!)^2(1/4)517=148896$ circular systems corresponding to $M=9$. For $M=13$, $s_4=48$ and we have 17 possibilities for (b_1, c_1, d_1), four of which admit no triad (a_2, a_3, a_4). Here we have 117 groups and $2(4!)^2(1/4)117=33696$ circular systems.

RESULT. *There are* 48 *true systems of magic circles of order* 3, 218592 *of order* 4, *and none of order* 2.

UNIVERSITY OF PORTO RICO

THE CONVERSE OF WARING'S PROBLEM

BY L. E. DICKSON

1. *Introduction.* In the most general Waring problem we are given a set of integers $\geqq 0$ and seek k such that every integer (or every sufficiently large integer) is a sum of k numbers of the set. We then call the set a k-set. In the converse problem, k is given and we seek all k-sets. There exist infinitely many k-sets; for example, (1) and (2).

In case every integer $\geqq 0$ is a sum of k numbers of a set, we call the latter a universal k-set. It must contain 0 and 1. By way of introduction, we construct some universal 2-sets.

I. As the nth element of the set choose the least integer which is not a sum of any two of the first $n-1$ elements. The set is composed of 0 and all positive odd integers.

II. The set with 0, 1, 2 and later elements chosen as in I is composed of 0, 1, $2+3x$, $(x=0, 1, \cdots)$. It is a universal 2-set.

III. After 0 and 1 choose the nth element as in I when n is odd, but subtract 1 from the least when n is even. We get the universal 2-set composed of 0, 1, $3+6x$, $4+6x$, $(x=0, 1, \cdots)$.

IV. After 0 and 1 employ blocks of three elements. Those in a block are odd (least as in I), odd (least), even (least less 1). We get the universal 2-set 0, 1, $3+10x$, $5+10x$, $6+10x$, $(x=0, 1, \cdots)$.

V. Our aim here is to construct a bizarre 2-set. After 0, 1 employ blocks of 2, 3, 4, $\cdots$ elements, where the last element of a block is even and the others are all odd, while the nth element is either the least or 1 less than the least integer which is not a sum of any two of the first $n-1$ elements. We get the set 0, 1, 3, 4, 9, 11, 16, 21, 23, 27, 28, 33, 35, 39, 41, 46, 53, 59, 65, 71, 77, 82, 83, 89, 95, 97, 101, 107, 114, 119, 125, 127, 133, 139, 145, 151, 156, 163, 169, 175, 181, 187, 193, 199, 205, 212, 217, 219, 225, 231, 237, 243, 249, 255, 261, 266, 267, 273, 279, 285, 291, 297, 303, 309, 311, 317, 322, 329, etc.

Unlike I–IV, there is apparently no simple independent definition of this 2-set. If we take the first elements of the successive blocks and form their differences of the second order,

we obtain a highly irregular sequence (including a negative number).

2. *Theorems regarding 2-Sets.* THEOREM 1. *If* $m \geqq 2$, $0 \leqq k \leqq m$, *the set*

$$(1)\quad 0, \cdots, k-1, k+1, \cdots, m-1, k+mx, \quad (x = 0, 1, \cdots),$$

is a universal 2-set.

The sums by 2 include $j+mx$, $(j=k, \cdots, k+m-1)$, which give all integers $\geqq k$. For $m=2$, $k=1$, we have I. If $m=2$, $k=0$ or 2, the set is 1 and all even integers $\geqq 0$.

If $k \leqq m-3$, we obtain a universal 2-set if we replace element $m-1$ in (1) by $2m-1$. If either $k \leqq m-2$ or $k \geqq 4$, we obtain a universal 2-set if we replace element 2 in (1) by $m+2$.

THEOREM 2. *If* $m \geqq j+1$, *we have the universal 2-set*

$$(2)\qquad 0, \cdots, m-1, m+j+mx, \qquad (x = 0, 1, \cdots).$$

If $j>0$, $m \geqq 2j+2$, the set (2) with element j deleted is a universal 2-set.

3. *Simplification of the Problem.* The problem to find all k-sets which involve a single linear function $n+mx$ reduces essentially* to the determination of integers $A, B, \cdots$, such that those sums by k of $n, A, B, \cdots$, which involve n, include a complete set of residues modulo m.

In case n is prime to m, we can find integers $a, b, \cdots$, such that $A \equiv na$, $B \equiv nb, \cdots$, (mod m). Then those sums by k of $1, a, b, \cdots$, which involve 1, include a complete set of residues modulo m. Hence if n is prime to m, the problem reduces to the case $n=1$.

4. *Determination of the 3-Sets.* We readily determine all 3-sets

$$(3)\qquad 0, a, b, 1+mx, \qquad (x = 0, 1, \cdots).$$

From the sums by three involving m we subtract 1 and get

$$(4)\qquad 0, 1, 2, a, b, a+1, b+1, 2a, a+b, 2b.$$

These ten numbers shall include a complete set of residues modulo m, whence $m \leqq 10$.

* Omitting the evaluation of the (small) limit beyond which every integer is a sum of k elements of the set.

First let m be even. Then the parity of a residue is invariant. The set (4) is symmetric in a and b. Hence the following two cases are exhaustive:

(5) if $a \equiv b \pmod 2$, exactly three of (4) are odd;

(6) if a is even, b odd, the odd (4) are $1, b, a+1, a+b$.

In any complete set of residues modulo m (m even), exactly half of the terms are odd. Hence $m \neq 10$.

Let $m = 8$. It suffices to treat case (6). The four odd in (6) must be congruent modulo 8 to 1, 3, 5, 7 in some order. By their sums, $a+b \equiv 3 \pmod 4$. If $b \equiv 3-a \pmod 8$, we see from (6) that $a+1$ and $3-a$ must be congruent to 5, 7 in some order, whence $a \equiv 4$ or 6. Finally, if $b \equiv 7-a$, then $a+1$, $7-a$ must be congruent to 3, 5, whence $a \equiv 2$ or 4.

THEOREM 3. *If m is even and >8, there is no 3-set of type* (3). *If $m = 8$,* (3) *is a 3-set if and only if a, b is one of the pairs* 2, 5; 3, 4; 4, 7; 5, 6.

Let $m = 9$. If $a \equiv 0$ or 1, there are two pairs of duplicates in (4). If $a \equiv 2$, the sum of the numbers in (4) with the duplicate a omitted is $13+5b$. But the sum of any complete set of residues modulo 9 is $\equiv 0$. Hence $b \equiv 1$ is a second duplicate. Similarly, $a \neq 5$ or 8. Hence a and by symmetry also b must be chosen from 3, 4, 6, 7.

Let $a = 3$. Then (4) are congruent to 0-4, 6, b, $b+1$, $b+3$, $2b$. If $b = 5$, $b+1 \equiv 6$, $2b \equiv 1$ give two pairs of duplicates. If $b = 6$, then $2b \equiv 3$. But $b = 4$ and 7 yield 3-sets.

THEOREM 4. *If $m = 9$,* (3) *is a 3-set if and only if a, b is one of the pairs* 3, 4; 3, 7; 4, 6; 6, 7.

5. *Cases when n and m are Even.* When n and m are even, the problem does not reduce to the case $n = 1$ as in §3. Consider a 3-set

$$0, a, b, 2 + mx, \qquad (x = 0, 1, \cdots). \tag{7}$$

From the sums by three involving m we subtract 2 and get

$$0, 2, 4, a, b, a + 2, b + 2, 2a, a + b, 2b. \tag{8}$$

These ten numbers must include a complete set of residues modulo m, whence $m \leqq 10$. Let m be 8 or 10. Evidently a and b

are not both even. If a is even and b odd, then b, $b+2$, $a+b$ are the only odd numbers in (8). Hence a and b are both odd. Then the only odd numbers (8) are

$$a, b, a+2, b+2. \tag{9}$$

Hence $m \neq 10$. Thus $m = 8$. Then (9) are congruent to 1, 3, 5, 7 in some order. By their sums, $a+b \equiv 2 \pmod 4$, whence $a \equiv b$. Hence $b \equiv a+4 \pmod 8$.

THEOREM 5. *If m is even and >8, there is no 3-set* (7). *If $m=8$,* (7) *is a 3-set if and only if a, b is one of the pairs* 1, 5 *or* 3, 7. *The same result holds if we replace $2+mx$ by $6+mx$ in* (7).

THE UNIVERSITY OF CHICAGO

HYPERTRANSCENDENTAL EXTENSIONS OF PARTIAL DIFFERENTIAL FIELDS*

BY H. W. RAUDENBUSH, JR.

1. *Introduction.* In the abstract or formal theory of ordinary algebraic differential equations,† the concept of *differential field* as defined by Baer‡ has a role analogous to that of field in abstract algebra.§ A differential field is a commutative field closed with respect to a formally defined operation called differentiation. The defining rules for differentiation are taken from the elementary properties of derivatives of functions of a single variable. In this paper, with an abstract theory of partial differential equations in mind, we define *partial differential fields*, selecting the defining rules for differentiation from the elementary properties of partial derivatives of functions of several variables.

* Presented to the Society, March 31, 1934.

† Raudenbush, *Differential fields and ideals of differential forms*, Annals of Mathematics, vol. 34 (1933), pp. 509–517; and *Ideal theory and algebraic differential equations*, Transactions of this Society, vol. 36 (1934), pp. 361–368. Also, O. Ore, *Formale Theorie der linearen Differentialgleichungen*, Journal für Mathematik, vol. 167 (1933), pp. 221–234, and vol. 168 (1934), pp. 233–252.

‡ R. Baer, *Algebraische Theorie der differentierbaren Funktionenkörper* I, Heidelberger Sitzungsberichte, 1927–28.

§ For the terms and theorems of abstract algebra, see van der Waerden, *Moderne Algebra*.

The object of this paper is to establish for partial differential fields analogs to Steinitz's theorems on transcendental extensions of fields. As in the case of differential fields,* the only considerable variation in the proofs from those in the algebraic theory is in the proof of the transitivity of dependence. Since dependence is here defined by means of partial differential equations, the necessary elimination is seriously complicated. This proof is given in full. The rest of the proofs may easily be supplied along the lines of the simpler theories.

2. *Definitions and Results.* Let m be a positive integer and $\mathfrak{F}$ a commutative field as in abstract algebra. We shall say that $\mathfrak{F}$ is a *partial differential field with m types of differentiation* if the following conditions hold:

(a) There is defined for each positive integer $i \leqq m$ an association of the elements of $\mathfrak{F}$ by which there corresponds to each element a of $\mathfrak{F}$ a unique element $\delta_i(a)$ called the *derivative of a of type i.* The operation of taking a derivative of type i is *called differentiation of type i.*

(b) For any elements a and b of $\mathfrak{F}$ and differentiation of any type $i \leqq m$,

$$\delta_i(a + b) = \delta_i(a) + \delta_i(b),$$
$$\delta_i(ab) = \delta_i(a)b + a\delta_i(b).$$

(c) For any element a of $\mathfrak{F}$ and differentiation of any types $i \leqq m$ and $j \leqq m$,

$$\delta_j(\delta_i(a)) = \delta_i(\delta_j(a)).$$

By an *extension* $\mathfrak{F}'$ of a partial differential field $\mathfrak{F}$ with m types of differentiation, we shall mean a partial differential field $\mathfrak{F}'$ with m types of differentiation which contains $\mathfrak{F}$ as a subset in such a way that sums, products, and derivatives of the various types of elements of $\mathfrak{F}$ are the same in $\mathfrak{F}'$.

By a *derivative* of an element a of a partial differential field with m types of differentiation, we shall mean any symbol $\delta_{j_1}(\delta_{j_2}(\delta_{j_3}(\cdots \delta_{j_n}(a) \cdots)))$, where n is a positive integer, and the j take on certain integral values from 1 to m. By (c), the order in which the j appear is immaterial. A polynomial with

* Raudenbush, *Differential fields*, loc. cit. References are given here for the algebraic theory.

coefficients in a partial differential field $\mathfrak{F}$ in elements of a subset A of an extension $\mathfrak{F}'$ of $\mathfrak{F}$ and their derivatives is called a *form in elements of A with coefficients in $\mathfrak{F}$.**

Let a be an element of an extension $\mathfrak{F}'$ of a partial differential field $\mathfrak{F}$. If a form in a with coefficients in $\mathfrak{F}$ not all zero is zero in $\mathfrak{F}'$, then we shall say that a is *algebraically transcendental with respect to $\mathfrak{F}$.*† If an element of $\mathfrak{F}'$ is not algebraically transcendental with respect to $\mathfrak{F}$, we shall say that it is *hypertranscendental with respect to $\mathfrak{F}$.*‡

If every element of an extension $\mathfrak{F}'$ of a partial differential field $\mathfrak{F}$ is algebraically transcendental with respect to $\mathfrak{F}$, then we shall say that $\mathfrak{F}'$ is an *algebraically transcendental extension of $\mathfrak{F}$.* If an extension $\mathfrak{F}'$ of a partial differential field $\mathfrak{F}$ is not an algebraically transcendental extension of $\mathfrak{F}$, we shall say that it is a *hypertranscendental extension of $\mathfrak{F}$.*

Let A be a set of elements of an extension $\mathfrak{F}'$ of a partial differential field $\mathfrak{F}$. Then the set of all quotients of forms in elements of A with coefficients in $\mathfrak{F}$, the denominators not zero, is an extension of $\mathfrak{F}$ and will be denoted by $\mathfrak{F}(A)$. Suppose that each element a_i of a set A of elements of $\mathfrak{F}'$ is hypertranscendental with respect to $\mathfrak{F}(A_i)$, where A_i is the set consisting of all elements of A except a_i. Then we shall say that A is *irreducible with respect to $\mathfrak{F}$* and that $\mathfrak{F}(A)$ is a *purely hypertranscendental extension of $\mathfrak{F}$.* We are now able to state the following results.

THEOREM 1. *A hypertranscendental extension $\mathfrak{F}'$ of a partial differential field $\mathfrak{F}$ contains a subset A irreducible with respect to $\mathfrak{F}$ such that $\mathfrak{F}'$ is an algebraically transcendental extension of the purely hypertranscendental extension $\mathfrak{F}(A)$ of $\mathfrak{F}$.*

The set A in the preceding result is not uniquely determined.

THEOREM 2. *If two sets B and C satisfy the conditions on A in the preceding theorem, then B and C have the same number (which may be transfinite) of elements.*

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* J. F. Ritt, *Differential Equations from the Algebraic Standpoint*, p. 157.

† This definition differs from that of A. Ostrowski, *Über Dirichletschen Reihen und algebraische Differentialgleichungen*, Mathematische Zeitschrift, vol. 8 (1920), pp. 241–298. According to Ostrowski, an element is algebraically transcendental if and only if it satisfies an algebraic Mayer system.

‡ The term "hypertranscendental", erroneously attributed to A. Bloch, is due to E. Maillet, Comptes Rendus, vol. 142 (1906), p. 829.

This number we call the *degree of hypertranscendency of $\mathfrak{J}'$ with respect to $\mathfrak{J}$.*

THEOREM 3. *If the degree of hypertranscendency of an extension $\mathfrak{J}'$ of a partial differential field $\mathfrak{J}$ is denoted by r and of an extension $\mathfrak{J}''$ of $\mathfrak{J}'$ with respect to $\mathfrak{J}'$ by s, then the degree of hypertranscendency of $\mathfrak{J}''$ with respect to $\mathfrak{J}$ is $r+s$.*

These theorems are seen to be analogs to Steinitz's theorems on degree of transcendency. As stated in §1 we omit the proofs except that of a lemma given in §3.

3. *The Fundamental Lemma.* This lemma, which is necessary to the results of §2, is essentially the analog to the theorem on the transitivity of algebraic dependence. Its proof, alone, of the proofs of the theorems of §2 and the lemmas necessary thereunto, presents new difficulties, hence it is given in full. The lemma is stated in terms of the concepts defined in this paper, rather than in terms of dependence.

LEMMA.* *Let $\mathfrak{J}'$ be an extension of a partial differential field $\mathfrak{J}$ with m types of differentiation. Let t be an element and U a set of elements such that t is algebraically transcendental with respect to $\mathfrak{J}(U)$ and each element of U algebraically transcendental with respect to $\mathfrak{J}$. Then t is algebraically transcendental with respect to $\mathfrak{J}$.*

Let $\delta_{j_1}(\delta_{j_2}(\delta_{j_3}(\cdots (\delta_{j_n}(a) \cdots)))$ be a derivative δa of an element a of $\mathfrak{J}'$. We shall say that n is the *total order* of δa. The number of j's that take the value of a certain $i \leqq m$ we call the *partial order* of δa *with respect to* i. We regard the symbol a as a derivative of the element a whose total and partial orders are all zero. Two derivatives are considered distinct if and only if there is a difference in their partial orders. It will not matter that distinct derivatives may represent elements equal in $\mathfrak{J}'$.

We proceed now to order the derivatives of any element a of $\mathfrak{J}'$. Let the derivatives $\delta^1 a$ and $\delta^2 a$ of a have k_i and l_i, respectively, as their partial orders with respect to $i \leqq m$. We shall say that $\delta^1 a$ is higher than $\delta^2 a$ if either of the following conditions holds.

(a) $\delta^1 a$ is of greater total order than $\delta^2 a$;

* For the proof of the corresponding lemma for differential fields, see Ostrowski, loc. cit., and Raudenbush, *Differential fields*, loc. cit. Our present lemma is different from Ostrowski's theorem, loc. cit., p. 278, for partial differentiation, since our definition of algebraically transcendental differs from his.

(b) $\delta^1 a$ is of the same total order as $\delta^2 a$ and the first difference $k_i - l_i$ that is distinct from zero is positive.

If $\delta^1 a$ is higher than $\delta^2 a$, we shall also say that $\delta^2 a$ is *lower* than $\delta^1 a$. Given two distinct derivatives, one must be higher than the other.

By the hypothesis of the lemma, there exists a form in t with coefficients in $\mathfrak{F}(U)$ not all zero which vanishes in $\mathfrak{F}'$. In the set Γ of all such forms there exists a form $G(t)$ having the following properties:

(a) The highest derivative $\delta^1 t$ effectively present in G is not higher than the highest derivative effectively present in any other form of Γ.

(b) The highest power of $\delta^1 t$ effectively present in G is not higher than the highest power of $\delta^1 t$ effectively present in any form of Γ that satisfies (a).

We shall assume, as we may, that G is written as a form in the elements of U. Now $S = \partial G/\partial(\delta^1 t) \neq 0$, since S either is of lower degree than G in $\delta^1 t$ or does not contain $\delta^1 t$. Let $\delta^2 t$ be $\delta_1(\delta^1 t)$, regarded as a derivative of t. Then

$$\delta^2 t = A/S, \tag{1}$$

where A and S are forms in t and the elements of U with coefficients in $\mathfrak{F}$ containing only derivatives of t which are lower than $\delta^2 t$.

The equation (1) involves a finite number of elements $u_1, \cdots, u_r$ of U and their derivatives. Proceeding as above, we may obtain for each u_j, $(j = 1, \cdots, r)$, a relation

$$\delta^1 u_j = A_j/S_j, \tag{2}$$

where A_j and $S_j \neq 0$ are forms in u_j with coefficients in $\mathfrak{F}$ containing only derivatives of u_j that are lower than the derivative $\delta^1 u_j$ of u_j.

If $\delta^1 u_j$ is lower than the highest derivative of u_j in (1), we differentiate the relation (2) until we obtain a relation

$$\delta^2 u_j = B_j/C_j, \qquad (j = 1, \cdots, r), \tag{3}$$

where $\delta^2 u_j$ is a derivative of u_j higher than any derivative of u_j in (1) and the forms B_j and $C_j \neq 0$ are forms in u_j containing only lower derivatives of u_j than $\delta^2 u_j$.

Let the total orders of $\delta^2 t$ and $\delta^2 u_j$ be, respectively, k and k_j. Let p be a positive integer and denote by $\delta^3 t$ and $\delta^3 u_j$ any derivatives of t and u_j which are also derivatives of $\delta^2 t$ and $\delta^2 u_j$ and which have total orders not greater than $k+p$ and k_j+p, respectively. By suitable differentiations, we obtain from (1) and (3) the equations

$$\delta^3 t = D/E, \qquad \delta^3 u_j = D_j/E_j, \qquad (j = 1, \cdots, r), \tag{4}$$

where D and $E \neq 0$ are forms in $t, u_1, \cdots, u_r$ containing no derivatives higher than $\delta^3 t$, and D_j and $E_j \neq 0$ are forms in u_j containing no derivatives higher than $\delta^3 u_j$. We assume that all such equations have been obtained. Starting with the highest, all derivatives of t appearing on the right sides of these equations which are also derivatives of $\delta^2 t$ and hence appear on the left sides may be successively removed from the right sides. Subsequently, again starting with the highest, all derivatives of u_j which are also derivatives of $\delta^2 u_j$ may be successively removed from the right sides. We thus obtain equations of the form

$$\delta^3 t = F/J \tag{5}$$

for every derivative $\delta^3 t$ of t which is also a derivative of $\delta^2 t$ and which is of total order not greater than $k+p$, where F and $J \neq 0$ are forms in $t, u_1, \cdots, u_r$ containing only such derivatives of t and u_j as are of total order not greater than $k+p$ and k_j+p, respectively, and which are not derivatives of $\delta^2 t$ and $\delta^2 u_j$, $(j=1, \cdots, r)$.

The number of such equations is

$$\alpha = \frac{1}{m!}(p+1)\cdots(p+m),$$

and the number of derivatives of $t, u_1, \cdots, u_r$ appearing in the right sides is

$$\beta \leqq \frac{1}{m!}\Big((p+k+1)\cdots(p+k+m) + \sum_{j=1}^{r}(p+k_j+1)\cdots(p+k_j+m) - (r+1)(p+1)\cdots(p+m)\Big).$$

Since the latter expression reduces to a polynomial of degree $m-1$ in p, we may fix upon a value for p large enough so that α exceeds β.

We now write the equations with a common denominator $P \neq 0$ on the right side

$$\delta^s t = Q_{s^s t}/P.$$

Regarding P and the Q as polynomials in the derivatives of t and $u_1, \cdots, u_r$, let n be the maximum of their degrees. Let γ be a positive integer to be fixed later. Form all power products V_i of total degree γ or less of the derivatives $\delta^s t$. We may write

$$V_i = R_i/P^\gamma,$$

where the degrees of the R_i regarded as polynomials in derivatives of $t, u_1, \cdots, u_r$ are at most $n\gamma$. We now regard the R_i as linear expressions in power products of the derivatives of $t, u_1, \cdots, u_r$. There are at most $((n\gamma+\beta) \cdots (n\gamma+1))/\beta!$ such power products in the linear expressions R_i. There are exactly $((\gamma+\alpha) \cdots (\gamma+1))/\alpha!$ distinct power products V_i and hence the same number of linear functions R_i. Hence we may choose γ sufficiently large so that the number of R_i exceeds the number of terms in the R_i. Since the coefficients of the R_i are in $\mathfrak{J}$ there exists a linear combination of the R_i with coefficients in $\mathfrak{J}$ not all zero which vanishes. The same linear combination of the V_i vanishes since $P \neq 0$. This is the form whose existence we need in order to prove the lemma.

BARNARD COLLEGE

ON A CLASS OF EXISTENCE THEOREMS IN DIFFERENTIAL GEOMETRY

BY T. Y. THOMAS AND JACK LEVINE

1. *Introduction.* Many problems in differential geometry require the integration of a mixed system of differential equations

$$(E_1) \qquad \frac{\partial u^\alpha}{\partial x^i} = \psi_i^\alpha(u, x),$$

$$(E_2) \qquad F_0(u, x) = 0,$$

which gives rise to an infinite sequence of sets of algebraic equations

$$(1) \qquad F_1 = 0, F_2 = 0, \cdots, F_N = 0, \cdots,$$

related to the system E by the following property.

A necessary and sufficient condition for the existence of a solution of the system E *is that there exist an integer* N *such that the first* N *sets of equations of the sequence* (1) *be algebraically consistent and that all their solutions satisfy the* $(N+1)$*st set of equations of the sequence.*

In many cases the functions ψ and F_0 are linear and homogeneous in the unknowns u with the result that the equations (1) are of a similar character; also the coefficients A of the unknowns in (1) are the components of tensor differential invariants. In fact, the left members of each set of the equations (1) break up into the components of scalars and tensors. We shall confine our attention to systems E of this type. Since such systems E always possess a trivial solution $u=0$, we shall mean by a solution $u(x)$ of E a non-trivial solution of this system.

A necessary and sufficient condition for the existence of a solution of the first N sets of equations of the sequence (1) is the vanishing of their resultant system $R_N(A)$. For equations of the type under consideration in which the coefficients are real quantities, the solution whose existence is implied by the vanishing of the resultant system will be real.*

* When the equations (1) are linear and homogeneous in the unknowns u^α, the resultant system R_N can be taken to consist of the totality of all determinants of order L, equal to the number of unknowns u^α, which can be formed from the matrix of the coefficients of the first N sets of equations (1).

By the above hypothesis concerning the tensor character of the equations (1), we can think of the A's as the coefficients of a set of multilinear forms Φ in one or more sets of cogredient and contragredient variables. Let $A \rightarrow \overline{A}$ as a result of a transformation of the variables of these forms. Then the vanishing of the resultants $R_N(A)$ implies the vanishing of the resultants $R_N(\overline{A})$ in the transformed coefficients $\overline{A}$, that is, the equations $R_N(A)=0$ are a set of *invariant equations* in the sense of the algebraic invariant theory. Now, by Gram's theorem,* the invariant equations $R_N(A)=0$ are equivalent to the identical vanishing of a set of covariants† of the forms Φ. Hence the conditions on the components A expressed by the equations $R_N(A)=0$ are equivalent to the vanishing of a set of tensor‡ differential invariants $T(A)$ with components which are polynomials in the coefficients A.

2. *The Resultant System.* Let $L(\geqq 2)$ denote the number of unknowns u in the system E, and let the symbol $\{F_N\}$ represent the first N sets of equations of the sequence (1). It will now be shown that the vanishing of the resultant system R_L is necessary and sufficient for E to admit a solution.

PROOF OF THE NECESSITY. If E has a solution, the set $\{F_L\}$ is satisfied by this solution, so that the resultant system R_L vanishes.

PROOF OF THE SUFFICIENCY. Suppose $R_L=0$. Then $\{F_L\}$ has a solution; in particular $\{F_1\}$ will have a solution. When the equations $F_2=0$ are added to the set $\{F_1\}$ two cases may arise:

(α) all solutions of $\{F_1\}$ satisfy $\{F_2\}$;

(β) the set $\{F_2\}$ is consistent but all solutions of $\{F_1\}$ do not satisfy $\{F_2\}$.

If (α) occurs, then E admits a solution by the existence theorem stated in §1. If (β) occurs, we add the equations $F_3=0$ to the set $\{F_2\}$ and consider cases analogous to the above, that is, cases (α) and (β) with $\{F_1\}$ replaced by $\{F_2\}$ and $\{F_2\}$ by $\{F_3\}$. Continuing in this manner, we must finally reach a set $\{F_M\}$ with

* See, for example, R. Weitzenböck, *Invariantentheorie*, 1923, p. 160.

† This is to be understood as including as a special case the vanishing of invariants of the forms Φ.

‡ This is considered to include the scalar differential invariants.

$M<L$ such that all solutions of $\{F_M\}$ satisfy $F_{M+1}=0$. In fact, either a set $\{F_M\}$ where $M<L-1$ is obtained such that all solutions of $\{F_M\}$ satisfy $F_{M+1}=0$, or else we arrive at the set $\{F_{L-1}\}$. But in this latter case there will be exactly $L-1$ independent equations in $\{F_{L-1}\}$, so that the equations $F_L=0$ cannot add new independent equations to the set $\{F_{L-1}\}$. Hence all solutions of $\{F_{L-1}\}$ will satisfy $F_L=0$, and the sufficiency is established.

3. *Fields of Parallel Vectors.* In the problem of determining the conditions under which a general affinely connected space admits a field of parallel contravariant vectors we arrive at the sequence

$$\lambda^i B^i_{jkl} = 0; \quad \lambda^i B^i_{jkl,m} = 0; \quad \lambda^i B^i_{jkl,m,p} = 0; \cdots$$

corresponding to (1), where B^i_{jkl} are the components of the curvature tensor B and $B^i_{jkl,m}$, $B^i_{jkl,m,p}$, $\cdots$ are the components of the successive covariant derivatives of B. Here $L=n$. Hence we have the following theorem.

THEOREM. *There exists a set of integral rational tensor differential invariants P of a general affinely connected space such that the vanishing of the set P is necessary and sufficient for the existence of a field of parallel contravariant vectors.*

It is to be noted that the order in the derivatives of the components of the affine connection of any invariant in P is $\leqq n$. Similar remarks apply to the existence of a field of parallel covariant vectors.

4. *First Integrals of the Paths of an Affinely Connected Space.* Consider an affinely connected space with symmetric affine connection Γ, and let us seek to determine a symmetric covariant tensor with components $a_{\alpha\beta\cdots\gamma}$ possessing the property that its covariant derivative vanishes, that is,

$$a_{\alpha\beta\cdots\gamma,\delta} = 0. \tag{2}$$

These equations correspond to the differential equations E_1; in this case the set E_2 does not appear. The number of unknowns $a_{\alpha\beta\cdots\gamma}$ in (2) is given by

$$L = \frac{(n+k-1)!}{(n-1)!k!},$$

where k is the number of indices in the set $\alpha\beta\cdots\gamma$.

THEOREM. *There exists a set of integral rational tensor differential invariants I_k whose vanishing is necessary and sufficient for* (2) *to admit a solution.*

The vanishing of the invariants I_k is a sufficient condition for the existence of a first integral,

$$a_{\alpha\beta\cdots\gamma}\frac{dx^\alpha}{ds}\cdot\frac{dx^\beta}{ds}\cdots\frac{dx^\gamma}{ds} = \text{const.}, \tag{3}$$

of the differential equations which define the paths of the affinely connected space. A necessary and sufficient condition for the existence of the first integral (3) is that

$$P(a_{\alpha\beta\cdots\gamma,\sigma}) = 0, \tag{4}$$

where P indicates the sum of the terms obtainable from the one in the parenthesis by a cyclic permutation of the indices $\alpha\beta\cdots\gamma\sigma$.

For linear first integrals we have the equations

$$a_{\alpha,\beta} + a_{\beta,\alpha} = 0. \tag{5}$$

Putting $a_{\alpha,\beta} = b_{\alpha\beta}$, we can deduce* as the system E_1:

$$\frac{\partial a_\alpha}{\partial x^\beta} = a_\sigma\Gamma^\sigma_{\alpha\beta} + b_{\alpha\beta}, \qquad b_{\alpha\beta,\gamma} = a_\sigma B^\sigma_{\gamma\alpha\beta}. \tag{6}$$

We obtain immediately from (5) the system E_2:

$$b_{\alpha\beta} + b_{\beta\alpha} = 0. \tag{7}$$

Corresponding to the equations (6) for the linear first integral, we have in the case of the quadratic first integral for the system E_1:

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* O. Veblen and T. Y. Thomas, Transactions of this Society, vol. 25 (1923), p. 592; also L. P. Eisenhart, *Non-Riemannian Geometry*, p. 120.

$$
\begin{aligned}
g_{ij,p} &= g_{ijp}, \\
g_{ijp,q} &= g_{ijpq} - g_{\alpha j}A^{\alpha}_{ipq} - g_{i\alpha}A^{\alpha}_{jpq}, \\
g_{ijpq,r} &= g_{\alpha\beta}G^{\alpha\beta}_{ijpqr} + g_{\alpha\beta\gamma}H^{\alpha\beta\gamma}_{ijpqr},
\end{aligned} \tag{8}
$$

where G and H are tensors; and for the system E_2:

$$
\begin{aligned}
&g_{ij} = g_{ji}, \quad g_{ijp} = g_{jip}, \quad g_{ijpq} = g_{jipq} = g_{ijpq}, \\
&g_{ijpq} + g_{jpiq} + g_{pijq} = 2(g_{i\alpha}A^{\alpha}_{jpq} + g_{j\alpha}A^{\alpha}_{piq} + g_{p\alpha}A^{\alpha}_{ijq}), \\
&g_{ijp} + g_{jpi} + g_{pij} = 0.
\end{aligned} \tag{9}
$$

A system E for first integrals (3) of the kth degree can be obtained analogous to the above systems E for linear and quadratic integrals. Hence we have the following result.

THEOREM. *There exists a set of integral rational tensor differential invariants I_k^* of a space with symmetric affine connection such that the vanishing of the set I_k^* is necessary and sufficient for the existence of a first integral of the kth degree.*

The number of unknowns L in the system (6) and (7) is equal to n^2+n. We can, however, eliminate the equations (7) from this system E by considering only the unknowns $b_{\alpha\beta}$ for which $\alpha>\beta$. For the equations (6), as so modified, the number of unknowns L is equal to $(n^2+n)/2$. Hence the maximum order of the differential invariants of a set $T(A)$ as predicted by the integer L according to the discussion of §2 need not necessarily be attained.

5. *Groups of Motions.* A metric space V_n will admit a group of motions with contravariant components ξ^α if, and only if, the equations of Killing are satisfied, that is,

$$
\xi_{\alpha,\beta} + \xi_{\beta,\alpha} = 0, \tag{10}
$$

where ξ_α is the associated covariant form of the components ξ^α. Since the equations (10) are of the same form as (5), it follows that the vanishing of the above set of differential invariants I_1^* for the space V_n is a necessary and sufficient condition for this space to admit a group of motions.

Now consider the sequence (1) constructed from the equations (6) and (7) with reference to the space V_n. If there exists an integer M such that the ranks of the matrices of the sets $\{F_M\}$ and $\{F_{M+1}\}$ are $L-r$, where $L=n^2+n$ denotes the number

of unknowns, then the V_n admits a group G_r of motions. Let $\{D_r\}$ represent the equations obtained by equating to zero all determinants of order $L-r+1$ in the matrix of the set $\{F_L\}$. The equations $\{D_r\}$ imply the vanishing of the resultant system R_L and hence by the argument of §2 there exists an integer M such that all solutions of $\{F_M\}$ satisfy $\{F_{M+1}\}$. But there must exist at least r independent solutions of $\{F_M\}$ in consequence of the conditions $\{D_r\}$. Hence the conditions $\{D_r\}$ are sufficient for the existence of a group of motions in r or more parameters. Evidently the conditions $\{D_r\}$ are necessary for the existence of such a group of motions. Moreover, the equations $\{D_r\}$ form an invariant system; this follows from the fact that $\{D_r\}$ assures the existence of $s \geqq r$ independent solutions of $\{F_L\}$ and hence of the transformed equations $\{\overline{F}_L\}$, which implies the existence of the corresponding equations $\{\overline{D}_r\}$. Hence there exists a set of integral rational tensor differential invariants H_r whose vanishing is equivalent to the conditions $\{D_r\}$.

A group of motions of a V_n involves at most $J=(n^2+n)/2$ parameters; also when V_n admits a G_J of motions the space is of constant curvature.* Hence the above considerations show the existence of J sets of differential invariants H, namely

$$H_1, H_2, \cdots, H_J, \tag{11}$$

such that the vanishing of any set of these invariants implies the vanishing of all preceding sets. We can now state the following theorem.

THEOREM. *The vanishing of the set of differential invariants H_r is a necessary and sufficient condition for a metric space V_n to admit a group of motions in R parameters, where $r \leqq R \leqq J$. Also a necessary and sufficient condition for the existence of a G_r of motions is the vanishing of the set H_r and non-vanishing of the set H_{r+1}.*

It is known that there exists no G_{J-1} of motions of a V_n.† This fact taken in connection with the above theorem shows that the vanishing of the set H_{J-1} implies the vanishing of the set H_J. Moreover, the conditions imposed by equating the invariants H_J to zero must be equivalent to the conditions

* See, for example, L. P. Eisenhart, *Riemannian Geometry*, 1926, p. 239.

† L. P. Eisenhart, loc. cit., p. 246.

$$B_{\alpha\beta\gamma\delta} = K(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}),$$

which characterize a space V_n of constant curvature K.

Theorems analogous to the above can be stated concerning the existence of affine and projective collineations in an affinely connected space.

6. *Metric Representation of an Affinely Connected Space.* It was shown in §4 that a necessary and sufficient condition for the system

$$\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} - g_{\sigma\beta}\Gamma^\sigma_{\alpha\gamma} - g_{\alpha\sigma}\Gamma^\sigma_{\beta\gamma} = 0 \tag{12}$$

to admit a solution in the symmetric unknowns $g_{\alpha\beta}$ is the vanishing of the set of invariants I_2. If the system (12) admits a solution $g_{\alpha\beta}(x)$ such that the determinant $|g_{\alpha\beta}|$ does not vanish identically, the affinely connected space S is said to reduce to a metric space based on the quadratic differential form

$$\phi = g_{\alpha\beta}dx^\alpha dx^\beta, \tag{13}$$

or in other words to admit an n-dimensional metric representation.

Now suppose that (12) possesses a solution $g_{\alpha\beta}$ such that the rank of the matrix $\|g_{\alpha\beta}\|$ is r, where $1 \leqq r < n$. Without loss of generality it can then be assumed that $\|g_{\alpha\beta}\|$ contains a non-vanishing determinant of order r in its upper left hand corner. Then

$$g_{ab} = g_{a\bar{b}}A^{\bar{a}}_{a}, \quad g_{a\beta} = g_{\alpha\beta}A^{\alpha}_{a}, \tag{14}$$

($a, b = 1, \cdots, r$; $\alpha, \beta = r+1, \cdots, n$). It has been shown that (13) can be reduced to the form

$$\sum_{a,b=1}^{r} h_{ab}(y)dy^a dy^b \tag{15}$$

by a coordinate transformation $x \to y$ if, and only if, the following equations are satisfied:*

* T. Y. Thomas, Proceedings of the National Academy of Sciences, vol. 20 (1934), pp. 215–219.

$$\frac{\partial A_\alpha^a}{\partial x^b} A_\beta^b - \frac{\partial A_\beta^a}{\partial x^b} A_\alpha^b - \frac{\partial A_\alpha^a}{\partial x^\beta} + \frac{\partial A_\beta^a}{\partial x^\alpha} = 0, \tag{16}$$

$(a, b = 1, \cdots, r; \alpha, \beta = r+1, \cdots, n)$. On substituting for the A's their values in terms of the $g_{\alpha\beta}$ as obtained from (14) and making use of (12), we find that the equations (16) are satisfied identically. Hence the transformation of (13) to the form (15) is possible.

Under the above hypothesis that the rank of $\|g_{\alpha\beta}\|$ is $r < n$, it is easily seen from (12) that

$$C_{\alpha k}^a(y) = 0, \tag{17}$$

$(a = 1, \cdots, r; \alpha = r+1, \cdots, n; k = 1, \cdots, n)$, where the C's are the components of the connection in the y coordinate system. It likewise follows from (12) and (17) that

$$\frac{\partial h_{ab}}{\partial y^\alpha} = 0, \tag{18}$$

$(a, b = 1, \cdots, r; \alpha = r+1, \cdots, n)$, that is, the coefficients h_{ab} in (15) depend on the variables $y^1, \cdots, y^r$ alone.

The correspondence

$$y^1, \cdots, y^r, y^{r+1}, \cdots, y^n \rightarrow y^1, \cdots, y^r$$

defines an isomorphism between the space S and an r-dimensional space S^*. Let the metric of S be defined by the form (13) and the metric of S^* by the form (15). Then the distance between any two points P and Q in S measured along a curve C joining these points is equal to the distance between the corresponding points P^* and Q^* measured along the corresponding curve C^* in S^*. The space S^* will therefore be said to be an r-dimensional metric representation of the affine space S.

THEOREM. *A necessary and sufficient condition for the space S to admit an r-dimensional metric representation S^*, where $1 \leqq r \leqq n$, is the vanishing of the set of differential invariants I_2.*

PRINCETON UNIVERSITY

THE RELATION BETWEEN LEWIS'S STRICT IMPLICATION AND BOOLEAN ALGEBRA†

BY E. V. HUNTINGTON

1. *Introduction.* The purpose of this note is to point out that the relation called "strict implication" in C. I. Lewis's system of logic can be shown to be substantially equivalent to the relation called subsumption ($p<q$) in ordinary Boolean algebra.

The proof hinges upon the establishment of two new theorems, numbered 23.1 and 23.2 below.

2. *Notation.* The principal symbols which occur in the formulas of Lewis's system are the following. [For convenience of printing, we shall use $+$, $'$, and $*$ in place of Lewis's "wedge" ($\vee$), "curl" ($\sim$), and "curl-diamond" ($\sim\diamond$); and to avoid confusion between Lewis's double use of the sign $=$, we shall replace one of these signs by ∞.]

(1). p, q, r, etc. are *variables*, elements of an undefined class K (interpretable as propositions).

(2). $p\times q$, or simply pq (read: p *times* q), is an object determined in an undefined way by the two elements p and q, and called their logical *product*.

(3). $p+q$ (read: p *plus* q) is an object determined in an undefined way by the two elements p and q, and called their logical *sum*.

(4). p' (read: p *prime*) is an object determined in an undefined way by the element p, and called the *contradictory* of p.

(5). $p\supset q$ (which may be read: p *hook* q) is also an object determined in an undefined way by the elements p and q. It may be called the "*implication* of p toward q."

(6). p^* (which may be read: p *star*) is an object determined in an undefined way by the element p. For lack of a better name, p^* may be called the "*ghost* of p."

(7). $p\infty q$ (which may be read: p *wave* q) is an object determined in an undefined way by the elements p and q. It may be called the "*equalization* of p and q."

Finally, the elements of the class K are classified into those

† Presented to the Society, June 20, 1934.

which are "accepted" in the system, and those that are not. To distinguish between these two classes, it will be convenient to introduce the following notation, which serves the same purpose as the Frege assertion sign.†

(8). "p in T" (read: *p is in T*), is an abbreviation for the assertion "p is an *accepted element* in the system."

On the basis of this notation, three *relations* may be defined as follows:

(9). "*p implies q*" is merely another way of saying "$(p \supset q)$ is in T."

(10). "*p is impossible*," or "*p is a ghost*," is merely another way of saying "p^* is in T."

(11). $p = q$ (read: "*p is equal to q*") is merely another way of saying "$(p \backsim\!\!\!\!\supset q)$ is in T."

3. *Rules of Procedure.* In every abstract deductive system, later formulas are derived from earlier ones in accordance with certain rules of procedure.

The rules of procedure adopted in Lewis's system are the following:

I. *Rule of inference.* Whenever in the course of the development of the system we find established the expression "p in T," and also the expression "$p \supset q$ in T," we may thereupon write down the expression "q in T." More briefly:

$$[(p \text{ in } T) \text{ and } (p \supset q \text{ in } T)] \rightarrow (q \text{ in } T),$$

where the "arrow" notation may be read: the antecedent "leads to" the consequent. Obviously, whenever $x \rightarrow y$ and $y \rightarrow z$, then $x \rightarrow z$.

II. *Rule of "replacement throughout."* If we find that an expression "$F(p, q, r, \cdots)$ in T" has been established, then we may replace p by any other element "throughout the expression" (that is, we must make the replacement simultaneously at every point where p occurs); similarly, we may replace q by any other element "throughout the expression"; etc. Here $F(p, q, r, \cdots)$

† This use of a "subclass T" was suggested by the present writer in 1933. See Transactions of this Society, vol. 35 (1933), p. 291, and this Bulletin, vol. 40 (1934), p. 127.

is any expression built up out of p, q, r, etc. by means of the operators of the system.

III. *Rule of "replacement by equals."* If we find that the expression "$p=q$" has been established, then at any point where p occurs in any formula of the system, we may replace p by q.

These two rules, taken together, are often called the rules of substitution.

IV. *Rule of "adjunction".*

$$[(p \text{ in } T) \text{ and } (q \text{ in } T)] \rightarrow (pq \text{ in } T).$$

V. *Rules of equivalence.*

(a) $\quad [(p \dashv q)(q \dashv p) \text{ in } T] \rightarrow (p = q)$; and

(b) $\quad (p = q) \rightarrow [(p \dashv q)(q \dashv p) \text{ in } T]$.

(c) $\quad [(p \dashv q) \text{ in } T] \text{ and } [(q \dashv p) \text{ in } T] \rightarrow (p = q)$.

Here (c) follows from (a) by aid of the rule of adjunction.

VI. Finally, we are to understand that whenever p and q are elements of K, the objects denoted by pq, $p+q$, p', $p \dashv q$, p^* and $p \backsim q$ are also elements of K.

4. *Established Theorems.* Starting from certain primitive propositions, or postulates, each of which is an expression of the form "$F(p, q, r, \cdots)$ in T," and employing the rules of procedure, Lewis establishes, among others, the following important theorems. [The references are to *Symbolic Logic* by Lewis and Langford (The Century Co., 1932).]

1. If p and q are in K, then pq is in K. [Page 123]
2. If p is in K, then p' is in K. [Page 123]
3. $pq=qp$. [12.15]
4. $(pq)r=p(qr)$. [12.5]

Def. $p+q=(p'q')'$. [11.01]

5. $pq+pq'=p$. [18.92]
6. $p \dashv p$ in T. [12.1]
7. $(p \dashv q) \dashv (q' \dashv p')$ in T. [12.43]
8. $(p \dashv p') \dashv (p \dashv q)$ in T. [18.12 with 19.74]
9. $(p \dashv q) = (pq')^*$. [11.02]
10. $p^* = p \dashv p'$. [18.12]

The following theorems 11–20 are deducible from 1–5 alone, since (by Rule III) 1–5 make the system a Boolean algebra.†

11. $p+q=(p'q')'$. [11.01]
12. $p(q+r)=pq+pr$. [16.72]
13. $p''=p$. [12.3]
14. $p=pp$. [12.7]
15. $pp'=qq'$.

Proof. By 13, 11, 5, 3, 4,

$$(pp')'=(p''p')'=p'+p=(p'q+p'q')+(pq+pq')$$
$$=q'p+q'p'+qp+qp',$$

and

$$(qq')'=(q'p+q'p')+(qp+qp').$$

Hence $pp'=qq'$, by 13.

16. Def. $Z=pp'$.
17. $pZ=Z$.

Proof. By 16, 4, 14, 16, $pZ=p(pp')=(pp)p'=pp'=Z$.

18. $Z+p=p$.

Proof. By 5, 14, 16 (and 11, 3), $p=pp+pp'=p+Z=Z+p$.

19. $Z=(pq)p'$.

Proof. By 17, 16, 4, 3, $Z=qZ=q(pp')=(qp)q'=(pq)p'$.

20. $pq'=p(pq)'$.

Proof. By 18, 16, 12, 11, 13,

$$pq'=Z+pq'=pp'+pq'=p(p'+q')=p(p''q'')'=p(pq)'.$$

From 9, by the aid of 19, we have also the interesting formula

21. $Z^*=(pq\dashv p)$.

Proof. By 19, $Z^*=[(pq)p']^*$, and by 9, $[(pq)p']^*=pq\dashv p$.

5. *New Theorems.* The following formulas, which are not explicitly mentioned in Lewis's book, are directly deducible from propositions 1–10. (The proof will be given below.)

23.1. $(p\dashv q)\dashv[(p\dashv pq)(pq\dashv p)]$ in T; and

23.2. $[(p\dashv pq)(pq\dashv p)]\dashv(p\dashv q)$ in T;

whence, by V(c) with II,

23.3. $(p\dashv q)=[(p\dashv pq)(pq\dashv p)]$.

† See Transactions of this Society, vol. 35 (1933), pp. 280–286, 557; or Mind, vol. 42 (1933), pp. 203–207.

Then from 23.3, by III,

$$[(p \dashv q) \text{ in } T] \rightarrow \{[(p \dashv pq)(pq \dashv p)] \text{ in } T\}.$$

But by V(a), $[(p \dashv pq)(pq \dashv p) \text{ in } T] \rightarrow (p = pq)$. Hence

24.1. $[(p \dashv q) \text{ in } T] \rightarrow (p = pq)$.

Again, from 23.3, by III,

$$\{[(p \dashv pq)(pq \dashv p)] \text{ in } T\} \rightarrow [(p \dashv q) \text{ in } T].$$

But by V(b), $(p = pq) \rightarrow [(p \dashv pq)(pq \dashv p) \text{ in } T]$. Hence

24.2. $(p = pq) \rightarrow [(p \dashv q) \text{ in } T]$.

These theorems 24.1 and 24.2 give us our main result as follows:

24.3. *Whenever we find the formula "$p \dashv q$" asserted, we may thereupon write down the formula "$p = pq$"; and conversely, whenever we find the formula "$p = pq$" established, we may write down that the formula "$p \dashv q$" is asserted.*

Now in Boolean algebra the relation of subsumption ($p < q$, read "p within q") is defined by the formulas

25.1. $(p < q) \rightarrow (p = pq)$, and $(p = pq) \rightarrow (p < q)$;

hence (in view of Rule III) 24.3 may be written as

25.2. $(p < q) \rightleftarrows [(p \dashv q) \text{ in } T]$.

Here $p < q$ is the Boolean relation of inclusion, and the "mutual arrow" notation, $x \rightleftarrows y$, means that each side "leads to" the other (that is, whenever we find x established, we may write down y, and whenever we find y established, we may write down x).

Finally, if the words "p implies q" are understood strictly in the sense defined in (9) [under "Notation" above] and if the words "p within q" are understood strictly in the sense defined in 25.1, we may write

25.3. (p implies q) $\rightleftarrows$ (p within q),

where the "mutual arrow" denotes inter-deducibility.

The significance of this result lies in the fact that the relation $p < q$ can be defined, and all its properties deduced, on the basis of the simple postulates for Boolean algebra (theorems 1–5

above), which involve only the familiar K, $+$, $\times$, $'$, without any reference to $p \dashv q$ or p^*; so that the elements $p \dashv q$ and p^* become in large measure superfluous.

It is true that the elements $p \dashv q$ and p^* may be interpreted arbitrarily in a variety of ways (without violating the postulates of the system), as shown by the examples in *Symbolic Logic*, Appendix II; and from a purely mathematical point of view these examples have decided interest. But from the point of view of practical logic, the statement "$p \dashv q$ is asserted" and the statement "p^* is asserted" would seem to be more important than the bare elements $p \dashv q$ and p^*.

The point is that, in Lewis's system, *no matter how the element* $p \dashv q$ *may be interpreted*, the statement "$p \dashv q$ is asserted" is always equivalent to the statement "p is within q" ($p < q$); and *no matter how the element* p^* *may be interpreted*, the statement "p^* is asserted" is always equivalent to the statement "$p =$ the zero element, pp', of the system."

In passing, one important feature of Lewis's system (not explicitly mentioned in his book) should be pointed out, namely, the fact that the "subclass T" may be chosen to a large extent arbitrarily. In fact, if K is finite, T may be a single element, or any power of 2 up to half of the whole class K.†

6. *Proofs of* 23.1 and 23.2. We now supply the proofs omitted above.

23.1. $(p \dashv q) \dashv [(p \dashv pq)(pq \dashv p)]$ *in* T.

Proof. By 10, 9, 8, $p^* \dashv (pq')^*$ in T.

By 7, $[p^* \dashv (pq')^*] \dashv [(pq')^{*\prime} \dashv p^{*\prime}]$ in T.

Hence by I, $(pq')^{*\prime} \dashv p^{*\prime}$ in T.

Hence by II, $[(pq')(pq')']^{*\prime} \dashv (pq')^{*\prime}$ in T,

whence by 16, $Z^{*\prime} \dashv (pq')^{*\prime}$ in T.

But by 7, 13, $[Z^{*\prime} \dashv (pq')^{*\prime}] \dashv [(pq')^* \dashv Z^*]$ in T.

Hence by I, $(pq')^* \dashv Z^*$ in T.

Hence by 9, $\{(pq')^* Z^{*\prime}\}^*$ in T.

Hence by 20, replacing p by $(pq')^*$ and q by Z^*, and using III, $\{(pq')^* [(pq')^* Z^*]'\}^*$ in T.

† Compare Mind, vol. 43 (1934), pp. 181–198, where, if the optional postulate 13 is omitted, the subclass A may be interpreted as a partial analog of the subclass T of the present paper.

Hence by 9, $(pq')^* \supset [(pq')^* Z^*]$ in T.
Hence by 20 and 19, $(pq')^* \supset \{[p(pq)']^*[(pq)p']^*\}$ in T.
Hence by 9, $(p \supset q) \supset [(p \supset pq)(pq \supset p)]$ in T.

23.2. $[(p \supset pq)(pq \supset p)] \supset (p \supset q)$ in T.

Proof. By 9 and 6, $(pp')^*$ in T, whence by 16, Z^* in T.
Hence by 19, when we replace p by $(pq')^*$ and q by Z^*,
$\{[(pq')^* Z^*](pq')^{*\prime}\}^*$ in T.
Hence by 9, $[(pq')^* Z^*] \supset (pq')^*$ in T.
Hence by 20 and 19, $\{[p(pq)']^*[(pq)p']^*\} \supset (pq')^*$ in T.
Hence by 9, $[(p \supset pq)(pq \supset p)] \supset (p \supset q)$ in T.

It will be noted that these proofs could be written out without the use of the "star" notation, since p^* serves merely as an abbreviation for $p \supset p'$, and $(pq')^*$ as an abbreviation for $p \supset q$.

HARVARD UNIVERSITY

ON INTEGRAL INVARIANTS OF NON-HOLONOMIC DYNAMICAL SYSTEMS†

BY A. E. TAYLOR

1. *Introduction.* It is well known that there are certain integral invariants associated with holonomic dynamical systems. Cartan‡ demonstrated that it is possible to characterize a Hamiltonian system by means of the relative integral invariant

$$\int_C \sum_i p_i dq_i - H dt.$$

The purpose of this paper is to extend the theory to the case of non-holonomic systems.

We shall adopt the following conventions in notation. There are three ranges of indices, which we shall usually represent by

† Presented to the Society, June, 20, 1934. I wish to acknowledge my indebtedness to A. D. Michal for criticism and suggestions during the writing of this paper.

‡ E. Cartan, *Leçons sur les Invariants Intégraux*, 1922, p. 13. Also W. F. Osgood, this Bulletin, vol. 39 (1933), p. 882, Abstract No. 343.

s, i, ν, $(s=1, 2, \cdots, n+k; i=1, 2, \cdots, n; \nu=1, 2, \cdots, k)$. Hereafter we shall not state explicitly these ranges for the indices.

2. *The Equations of Motion.* Consider a non-holonomic dynamical system with n degrees of freedom and $n+k$ natural coordinates $q_1, \cdots, q_{n+k}$, subject to the non-integrable relations

$$\dot{q}_{n+\nu} = \sum_{i=1}^{n} a_{\nu i}\dot{q}_i + a_\nu, \tag{1}$$

where the a's are continuous functions of $q_1, \cdots, q_{n+k}, t$, possessing continuous first partial derivatives. For such a system, assumed conservative, the equations of motion are*

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} &= \frac{\partial L}{\partial q_i} - \sum_{\nu=1}^{k} a_{\nu i}\lambda_\nu, \qquad (i = 1, \cdots, n), \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_{n+\nu}} &= \frac{\partial L}{\partial q_{n+\nu}} + \lambda_\nu, \qquad (\nu = 1, \cdots, k), \end{aligned} \tag{2}$$

where the λ's are Lagrangian multipliers, functions of the time. It is to be observed that in case the system is holonomic all the λ's are zero, and Lagrange's equations hold in the usual form.

If we introduce new coordinates (q_s, p_s, t), where $p_s = \partial L/\partial \dot{q}_s$, and set

$$H = \sum_{s=1}^{n+k} p_s\dot{q}_s - L, \tag{3}$$

then equations (2) become†

$$\begin{aligned} \frac{dq_s}{dt} &= \frac{\partial H}{\partial p_s}, \\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial q_i} - \sum_{\nu=1}^{k} a_{\nu i}\lambda_\nu, \\ \frac{dp_{n+\nu}}{dt} &= -\frac{\partial H}{\partial q_{n+\nu}} + \lambda_\nu, \end{aligned} \tag{4}$$

* P. Woronetz, *Über die Bewegung eines starren Körpers*, Mathematische Annalen, (1911), p. 421; also Whittaker, *Analytical Dynamics*, 1927, p. 214.

† For the transformation process, see Appell, *Traité de Mécanique Rationnelle*, vol. 2, 1911, p. 403.

and we find, in addition, that

$$(5) \qquad \frac{dH}{dt} = \sum_{\nu=1}^{k} a_\nu \lambda_\nu - \frac{\partial L}{\partial t}, \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

3. *Integral Invariants.** Consider a system of differential equations

$$(6) \qquad \frac{dx_i}{dt} = X_i(x_1, \cdots, x_n, t),$$

where the X's are continuous, together with all of their partial derivatives of the first order, in the neighborhood of the point $(x_1^0, \cdots, x_n^0, 0)$, and where not all the X's vanish at this point. Then we can write the solution in the form

$$(7) \qquad x_i = f_i(t; x^0, \cdots, x_n^0).$$

Let the initial values be made to depend upon a parameter:

$$x_i^0 = x_i^0(\alpha), \qquad (\alpha_0 \leqq \alpha \leqq \alpha_1),$$

these functions being continuous, with continuous first derivatives, such that

$$\sum_{i=1}^{n} \left(\frac{dx_i^0}{d\alpha}\right)^2 > 0.$$

Putting these values in (7), we get equations of the form

$$x_i = F_i(t, \alpha) = f_i[t; x_1^0(\alpha), \cdots, x_n^0(\alpha)],$$

and these equations represent a *tube* of trajectories of the given system (6). If we envisage a regular curve C encircling this tube, in such manner that each trajectory cuts C once and only once, in a point for which $t>0$, then we may speak of C as a *simple, closed circuit* of the tube of trajectories.

Now let functions $A_i(x_1, \cdots, x_n, t)$ and $A(x_1, \cdots, x_n, t)$ be given, and consider

$$(8) \qquad \int_C \sum_{i=1}^{n} A_i dx_i + A dt \equiv \int_{\alpha_0}^{\alpha_1} \left[\sum_{i=1}^{n} A_i \frac{\partial x_i}{\partial \alpha} + A \frac{\partial t}{\partial \alpha}\right] d\alpha.$$

If this integral has the same value for all such simple closed

* For other forms of definition, see Appell, loc. cit., vol. 2, p. 464.

circuits C encircling an arbitrarily chosen *tube* of trajectories of (6), it is called a *relative* integral invariant of the system (6). If, in particular, the circuit corresponds to a fixed instant t, then the integral has the form

$$\int_C \sum_{i=1}^{n} A_i dx_i \equiv \int_{\alpha_0}^{\alpha_1} \sum_{i=1}^{n} A_i \frac{\partial x_i}{\partial \alpha} d\alpha.$$

This we shall call a Poincaré* integral invariant, while (8) will be referred to as a Cartan invariant.

4. *The Integral Invariant.* Let there be given a non-holonomic system, subject to the kinematical conditions (1), and having the equations of motion (2). These two sets of equations define the q's and λ's as functions of the time. There is a $2(n+k)$-parameter family of path curves of the system, represented by the functions

$$q_s = q_s(t; q_1^0, \cdots, q_{n+k}^0; \dot{q}_1^0, \cdots, \dot{q}_{n+k}^0), \tag{9}$$

where the $2(n+k)$ constants $(q^0, \dot{q}^0)$ are given initial values.

Let these path curves be interpreted in the space of the variables $(q_1, \cdots, q_{n+k}, \dot{q}_1, \cdots, \dot{q}_{n+k}, t)$ and consider an arbitrary regular curve† C_0 in the hyperplane $t=0$:

$$C_0: \qquad q_s^0 = q_s^0(\alpha), \qquad \dot{q}_s^0 = \dot{q}_s^0(\alpha), \qquad (\alpha_0 \leqq \alpha \leqq \alpha_1).$$

The tube of path curves emanating from the curve C_0 is the locus defined by the equations

$$q_s = q_s(t, \alpha), \qquad \dot{q}_s = \dot{q}_s(t, \alpha), \tag{10}$$

obtained from (9) by means of the equations of C_0.

Now consider the action integral in the form

$$J = \int_0^{t_1} L dt, \tag{11}$$

where t_1 is a variable upper limit. When this integral is extended along a path curve on the tube, it becomes a function of α. Let the aggregate of values of t_1 correspond to a simple closed circuit C of the tube. We introduce a new variable u, so that if

* H. Poincaré, *Méthodes Nouvelles de la Mécanique Céleste*, 1899, vol. 3, p. 4.

† It is sufficient to require that these functions be continuous, admitting continuous first derivatives, not all of which vanish simultaneously.

$$t = t(u, \alpha), \tag{12a}$$

then

$$t(u_0, \alpha) = 0, \quad t(u_1, \alpha) = t_1, \quad \frac{dt}{du} = \frac{\partial t(u, \alpha)}{\partial u} = \rho > 0, \tag{12b}$$

for all values of α, where u_1 and u_0 are constants. This can be done in a variety of ways, so that ρ may be taken arbitrarily,* within certain limits.

The integral (11) now becomes

$$J(\alpha) = \int_{u_0}^{u_1} F du, \tag{13}$$

where

$$F(q_1, \cdots, q_{n+k}; q_1', \cdots, q_{n+k}', t', t) = L\left(q_1, \cdots, q_{n+k}; \frac{q_1'}{t'}, \cdots, \frac{q_{n+k}'}{t'}, t\right)t',$$

$$t' = \frac{dt}{du}, \qquad q_s' = \frac{\partial q_s}{\partial u} = \dot{q}_s t'.$$

We next compute† $J'(\alpha)$, using Leibniz's rule, which is permissible since we have all the desired continuity. Upon carrying out the differentiation, and integrating by parts, we find

$$J'(\alpha) = \int_{u_0}^{u_1} \rho\left[\sum_{s=1}^{n+k}\left\{\frac{\partial L}{\partial q_s} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s}\right\}\frac{\partial q_s}{\partial \alpha} + \left\{\frac{\partial L}{\partial t} - \frac{d}{dt}\left[L - \sum_{s=1}^{n+k}\frac{\partial L}{\partial \dot{q}_s}\dot{q}_s\right]\right\}\frac{\partial t}{\partial \alpha}\right] du + \left[\sum_{s=1}^{n+k}\frac{\partial L}{\partial \dot{q}_s}\frac{\partial q_s}{\partial \alpha} + \left(L - \sum_{s=1}^{n+k}\frac{\partial L}{\partial \dot{q}_s}\dot{q}_s\right)\frac{\partial t}{\partial \alpha}\right]_{u_0}^{u_1}.$$

Using (2), (3), and (5), we obtain from this

* It is sufficient to require that $\rho = \rho(t, \alpha)$ be a positive integrable function of t for every α.

† The prime denotes differentiation with respect to α.

$$J'(\alpha) = \int_{u_0}^{u_1} \rho \left[\sum_{r=1}^{k} \left\{ \sum_{i=1}^{n} a_{ri} \frac{\partial q_i}{\partial \alpha} + a_r \frac{\partial t}{\partial \alpha} - \frac{\partial q_{n+r}}{\partial \alpha} \right\} \lambda_r \right] du + \left[\sum_{s=1}^{n+k} p_s \frac{\partial q_s}{\partial \alpha} - H \frac{\partial t}{\partial \alpha} \right]_{u_0}^{u_1}. \tag{14}$$

Let us now integrate $J'(\alpha)$ around the curve C determined by $u = u_1$:

$$\int_C J'(\alpha) d\alpha \equiv \int_{\alpha_0}^{\alpha_1} J'(\alpha) d\alpha = J(\alpha_1) - J(\alpha_0) = 0,$$

since α_0 and α_1 correspond to the same points on the closed curve. Thus we have

$$\int_{u_0}^{\alpha_1} d\alpha \int_{u_0}^{u_1} \rho \left[\sum_{r=1}^{k} \left\{ \sum_{i=1}^{n} a_{ri} \frac{\partial q_i}{\partial \alpha} + a_r \frac{\partial t}{\partial \alpha} - \frac{\partial q_{n+r}}{\partial \alpha} \right\} \lambda_r \right] du + \int_{\alpha_0}^{\alpha_1} \left[\sum_{s=1}^{n+k} p_s \frac{\partial q_s}{\partial \alpha} - H \frac{\partial t}{\partial \alpha} \right] d\alpha = \int_{\alpha_0}^{\alpha_1} \sum_{s=1}^{n+k} p_s^0 \frac{\partial q_s^0}{\partial \alpha} d\alpha. \tag{15}$$

The integral on the right is taken around the curve C_0, and that on the left around the simple closed circuit C. Since the right side is independent of the circuit C, the expression on the left of (15) is an integral invariant of the Cartan type, as defined in §3. In case the system is holonomic, we get the usual Cartan invariant (see Introduction). Equation (15) includes as a special case the Poincaré invariant

$$\int_{\alpha_0}^{\alpha_1} d\alpha \int_0^{t_1} \left[\sum_{r=1}^{k} \left\{ \sum_{i=1}^{n} a_{ri} \frac{\partial q_i}{\partial \alpha} - \frac{\partial q_{n+r}}{\partial \alpha} \right\} \lambda_r \right] dt + \int_{\alpha_0}^{\alpha_1} \sum_{s=1}^{n+k} p_s \frac{\partial q_s}{\partial \alpha} d\alpha, \tag{16}$$

when t_1 is a fixed upper limit. These results may be stated in the following theorem.

THEOREM. *The non-holonomic dynamical system defined by* (1) *and* (2) *admits the relative integral invariant* (15).

5. *A Dynamical Theorem.* We have seen that the equations of motion of a non-holonomic system can be written in the Hamil-

tonian form (4). Suppose that we consider an arbitrary system of differential equations

$$\frac{dq_s}{dt} = Q_s, \qquad \frac{dp_s}{dt} = P_s, \tag{17}$$

where the Q's and P's are any functions of (q_s, p_s, t) with the desired continuity. Let $a_{\nu i}(q, t)$, $a_\nu(q, t)$, $\lambda_\nu(t)$, and $H(q, p, t)$ be given functions, with continuity as required, and suppose that

$$\int_{\alpha_0}^{\alpha_1} d\alpha \int_{u_0}^{u_1} \rho\left[\sum_{\nu=1}^{k}\left\{\sum_{i=1}^{n} a_{\nu i}\frac{\partial q_i}{\partial \alpha} + a_\nu\frac{\partial t}{\partial \alpha} - \frac{\partial q_{n+\nu}}{\partial \alpha}\right\}\lambda_\nu\right] du + \int_{\alpha_0}^{\alpha_1}\left[\sum_{s=1}^{n+k} p_s\frac{\partial q_s}{\partial \alpha} - H\frac{\partial t}{\partial \alpha}\right] d\alpha \tag{18}$$

is a Cartan integral invariant of the system (17). This means that $u = u_1$ determines a certain simple closed circuit C on the tube of trajectories of (17), and that the integral (18) is independent of u_1. On differentiating with respect to u_1, and dropping the subscript, we have

$$\int_{\alpha_0}^{\alpha_1} \rho\left[\sum_{\nu=1}^{k}\left\{\sum_{i=1}^{n} a_{\nu i}\frac{\partial q_i}{\partial \alpha} + a_\nu\frac{\partial t}{\partial \alpha} - \frac{\partial q_{n+\nu}}{\partial \alpha}\right\}\lambda_\nu + \sum_{s=1}^{n+k}\left\{\frac{dp_s}{dt}\frac{\partial q_s}{\partial \alpha} + p_s\frac{d}{dt}\frac{\partial q_s}{\partial \alpha}\right\} - \frac{dH}{dt}\frac{\partial t}{\partial \alpha} - H\frac{d}{dt}\frac{\partial t}{\partial \alpha}\right] d\alpha = 0.$$

Integrating by parts, we find

$$\int_{\alpha_0}^{\alpha_1} \rho\left[\sum_{i=1}^{n}\left\{\frac{dp_i}{dt} + \frac{\partial H}{\partial q_i} + \sum_{\nu=1}^{k} a_{\nu i}\lambda_\nu\right\}\frac{\partial q_i}{\partial \alpha} + \sum_{\nu=1}^{k}\left\{\frac{dp_{n+\nu}}{dt} + \frac{\partial H}{\partial q_{n+\nu}} - \lambda_\nu\right\}\frac{\partial q_{n+\nu}}{\partial \alpha} + \sum_{s=1}^{n+k}\left\{-\frac{dq_s}{dt} + \frac{\partial H}{\partial p_s}\right\}\frac{\partial p_s}{\partial \alpha} + \left\{\sum_{\nu=1}^{k} a_\nu\lambda_\nu - \frac{dH}{dt} + \frac{\partial H}{\partial t}\right\}\frac{\partial t}{\partial \alpha}\right] d\alpha = 0,$$

or

$$\int_C \rho\Big[\sum_{i=1}^{n}\Big\{\frac{dp_i}{dt}+\frac{\partial H}{\partial q_i}+\sum_{\nu=1}^{k}a_{\nu i}\lambda_\nu\Big\}dq_i$$
$$+\sum_{\nu=1}^{k}\Big\{\frac{dp_{n+\nu}}{dt}+\frac{\partial H}{\partial q_{n+\nu}}-\lambda_\nu\Big\}dq_{n+\nu}$$
$$(19)\qquad +\sum_{s=1}^{n+k}\Big\{-\frac{dq_s}{dt}+\frac{\partial H}{\partial p_s}\Big\}dp_s$$
$$+\Big\{\sum_{\nu=1}^{k}a_\nu\lambda_\nu-\frac{dH}{dt}+\frac{\partial H}{\partial t}\Big\}dt\Big]=0.$$

But ρ is a positive function, and, from the way in which it was introduced (see (12b)), it is arbitrary. Hence we may infer that each parenthesis vanishes separately. The resulting equations are precisely (4) and (5), those of a non-holonomic Hamiltonian system. Accordingly we have the following result.

THEOREM. *If the integral* (18) *is a Cartan integral invariant of the system of differential equations* (17), *for a given set of* λ'*s*, $a_{\nu i}$'*s*, *and* a_ν'*s*, *and a given function* H, *then the system* (17) *is of the form* (4), *and consequently there is a non-holonomic dynamical system for which* (17) *are the equations of motion.*

The result is that the non-holonomic system is completely characterized by the integral invariant (18), which may be written more suggestively in the form

$$\int_C\int_0^t\sum_{\nu=1}^{k}\lambda_\nu\Big\{\sum_{i=1}^{n}a_{\nu i}dq_i+a_\nu dt-dq_{n+\nu}\Big\}dt$$
$$+\int_C\sum_{s=1}^{n+k}p_s dq_s-Hdt,$$

it being understood that the upper limit t is a function of α determined by the curve C.

CALIFORNIA INSTITUTE OF TECHNOLOGY

THE SUMMATION OF SERIES OF ORTHOGONAL POLYNOMIALS*

BY DUNHAM JACKSON

1. *Introduction.* The convergence of series of orthogonal polynomials can be treated under very broad hypotheses as to the character of the weight function with respect to which the property of orthogonality holds, if compensating restrictions are placed on the function whose series development is under consideration.† It is to be shown here that certain questions of summability lend themselves to a comparable treatment, simple as to demonstration and in some ways perhaps crude as to results, but characterized by broad generality in the hypotheses on the weight function. The discussion of summability, unlike the particular theory of convergence to which reference has been made, depends on the representation of the partial sum of the series by means of the Christoffel-Darboux identity. The method will be illustrated first under simple conditions, which will then be seen to admit generalization in various directions.

The problem of trigonometric approximation, as usual, is similar in principle but somewhat different in detail.

2. *Application of Schwarz's Inequality to the First Arithmetic Mean.* Let $p_0(x), p_1(x), \cdots$ be the system of normalized orthogonal polynomials for the interval $(-1, 1)$ corresponding to the summable non-negative weight function $\rho(x)$, let a function $f(x)$ (always supposed to be such that ρf is summable, and to be subjected to further restrictions later) be expanded in series of these polynomials, and let $s_n(x)$ be the partial sum of the series through terms of the nth degree. Let

$$\sigma_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} s_k(x).$$

The sums $s_k(x)$ and the corresponding errors of approximation $s_k(x) - f(x)$ can be represented in compact form by use of the

* Presented to the Society, April 7, 1934.

† See, for example, the writer's paper entitled *Certain problems of closest approximation*, this Bulletin, vol. 39 (1933), pp. 889–906.

Christoffel-Darboux identity,* and this representation gives for the error of the arithmetic mean the expression

$$\sigma_n(x) - f(x) = \frac{1}{n}\int_{-1}^{1} \rho(t)[f(t) - f(x)] \times \sum_{k=0}^{n-1} \frac{a_k}{a_{k+1}} \cdot \frac{p_{k+1}(t)p_k(x) - p_k(t)p_{k+1}(x)}{t - x}\, dt,$$

where $a_k > 0$ is the leading coefficient in the normalized polynomial $p_k(x)$. Let the difference quotient $[f(t) - f(x)]/(t - x)$, for any specified value of x, be denoted by $\phi(t)$, and let

$$H_n(x, t) = \sum_{k=0}^{n-1} \frac{a_k}{a_{k+1}} p_{k+1}(t)p_k(x).$$

Then, if $\rho\phi$ is summable,

$$n[\sigma_n(x) - f(x)] = \int_{-1}^{1} \rho(t)\phi(t)H_n(x, t)dt - \int_{-1}^{1} \rho(t)\phi(t)H_n(t, x)dt. \tag{1}$$

Under the assumption that $\rho(t)[\phi(t)]^2$ *is summable*, let Schwarz's inequality be applied to the first integral on the right, the integrand being regarded as the product of the factors $[\rho(t)]^{1/2}\phi(t)$ and $[\rho(t)]^{1/2}H_n(x, t)$. It appears that

$$\left[\int_{-1}^{1} \rho(t)\phi(t)H_n(x, t)dt\right]^2 \leqq \int_{-1}^{1} \rho(t)[\phi(t)]^2 dt \int_{-1}^{1} \rho(t)[H_n(x, t)]^2 dt. \tag{2}$$

By the property of orthogonality of the p's,

$$\int_{-1}^{1} \rho(t)[H_n(x, t)]^2 dt = \sum_{k=0}^{n-1} [(a_k/a_{k+1})p_k(x)]^2.$$

It is known† that $a_k/a_{k+1} < 1$, and hence, in terms of the notation

* See, for example, D. Jackson, *Series of orthogonal polynomials*, Annals of Mathematics, (2), vol. 34 (1933), pp. 527–545, p. 531.

† See Annals of Mathematics, loc. cit., pp. 531–532.

$$K_n(x, t) = p_0(x)p_0(t) + p_1(x)p_1(t) + \cdots + p_n(x)p_n(t),$$

$$\int_{-1}^{1} \rho(t)[H_n(x, t)]^2 dt \leqq K_n(x, x). \tag{3}$$

By the property of orthogonality once more,

$$\int_{-1}^{1} \rho(t)[K_n(x, t)]^2 dt = \sum_{k=0}^{n} [p_k(x)]^2 = K_n(x, x).$$

If $\rho(t)$ has a positive lower bound, say $\rho(t) \geqq v > 0$ for $-1 \leqq t \leqq 1$,

$$\int_{-1}^{1} [K_n(x, t)]^2 dt \leqq \frac{1}{v} \int_{-1}^{1} \rho(t)[K_n(x, t)]^2 dt = \frac{1}{v} K_n(x, x). \tag{4}$$

A lemma* giving a relation of inequality between an integral involving a polynomial and the maximum of the absolute value of the polynomial can be stated in a special case as follows. *If $P_n(t)$ is a polynomial of the nth degree, and if*

$$H_{n2} = \int_{-1}^{1} [P_n(t)]^2 dt,$$

then

$$| P_n(t) | \leqq \frac{2[2(n + 1)H_{n2}]^{1/2}}{(1 - t^2)^{1/2}}$$

for $-1 < t < 1$. As $K_n(x, t)$ is for any fixed value of x a polynomial of the nth degree in t, this lemma may be combined with the relation (4) to give

$$| K_n(x, t) | \leqq \frac{2[2(n + 1)(1/v)K_n(x, x)]^{1/2}}{(1 - t^2)^{1/2}},$$

and in particular†

$$K_n(x, x) \leqq \frac{2[(2/v)(n + 1)K_n(x, x)]^{1/2}}{(1 - x^2)^{1/2}},$$

* See this Bulletin, loc. cit., p. 897, Lemma 4.

† See also J. A. Shohat, *On the development of continuous functions in series of Tchebycheff polynomials*, Transactions of this Society, vol. 27 (1925), pp. 537–550; G. Szegö, *Über den asymptotischen Ausdruck von Polynomen, die durch eine Orthogonalitätseigenschaft definiert sind*, Mathematische Annalen, vol. 86 (1922), pp. 114–139, p. 129.

$$(5)\quad [K_n(x,x)]^{1/2} \leqq \frac{2[(2/v)(n+1)]^{1/2}}{(1-x^2)^{1/2}}, \quad K_n(x,x) \leqq \frac{8(n+1)}{v(1-x^2)}.$$

Hence, in view of (2) and (3), the first integral in the right-hand member of (1) does not exceed a constant multiple of $n^{1/2}$. Similar analysis applies to the second integral, since

$$\int_{-1}^{1} \rho(t)[H_n(t,x)]^2 dt = \sum_{k=0}^{n-1} [(a_k/a_{k+1})p_{k+1}(x)]^2 \leqq K_n(x,x).$$

If $\rho\phi^2$ is summable, therefore, while ρ has a positive lower bound, $\sigma_n(x)$ converges toward $f(x)$ as n becomes infinite, x being interior to the interval $(-1, 1)$, and $|\sigma_n(x)-f(x)|$ does not exceed a quantity of the order of $n^{-1/2}$. If $\int\rho\phi^2$ as a function of x is bounded over a set of points contained in a closed interval interior to $(-1, 1)$, the convergence is uniform over that point set.

A sufficient condition* for convergence of $s_n(x)$ toward $f(x)$ in the interior of the interval is that $\lim_{\delta=0}\omega(\delta)/\delta^{1/2}=0$, where $\omega(\delta)$ is the modulus of continuity of $f(x)$ for $-1\leqq x\leqq 1$. A first comparison shows the new condition for convergence of $\sigma_n(x)$ in its most unfavorable light; for convergence of $s_n(x)$ of course implies convergence of $\sigma_n(x)$, and the hypothesis that $\omega(\delta)/\delta^{1/2}\to 0$ does not require that $\rho\phi^2$ be summable, even if ρ itself is bounded. The following points are to be emphasized, however, on behalf of the new method.

(a) The theorem on convergence of $s_n(x)$ does not give the information that has been obtained by the new method as to the order of magnitude of $\sigma_n(x)-f(x)$.

(b) The hypothesis on $\omega(\delta)$ demands that $f(x)$ be continuous and subject to further restriction throughout the entire interval† $(-1, 1)$, while the summability of $\rho\phi^2$ for a particular x does not require that $f(x)$ be continuous or even bounded elsewhere.

(c) For mere convergence of $\sigma_n(x)$, apart from degree of convergence, summability of $\rho\phi^2$ can readily be replaced by a less stringent requirement, as will be shown in the next section.

* This Bulletin, loc. cit., p. 906, Theorem 15.

† It is obvious, but irrelevant for the comparison, that for a more general form of statement the definition of $f(x)$ at a set of points of measure zero is immaterial.

(d) The convergence theorem can also be generalized in other ways, as will be pointed out later.

3. *More General Function $f(x)$.* Let it be supposed now that $\rho(t)\,|\phi(t)|^\alpha$ is summable, where α has a value belonging to the range $1<\alpha<2$. With reference to the first integral on the right of (1) once more, let Hölder's inequality be applied to the product of the factors $[\rho(t)]^{1/\alpha}|\phi(t)|$ and $[\rho(t)]^{(\alpha-1)/\alpha}|H_n(x,t)|$:

$$\left|\int_{-1}^{1}\rho(t)\phi(t)H_n(x,t)dt\right| \leqq \int_{-1}^{1}\rho(t)\,|\,\phi(t)H_n(x,t)\,|\,dt \tag{6}$$
$$\leqq \left[\int_{-1}^{1}\rho(t)\,|\,\phi(t)\,|^\alpha dt\right]^{1/\alpha}\left[\int_{-1}^{1}\rho(t)\,|\,H_n(x,t)\,|^{\alpha/(\alpha-1)}dt\right]^{(\alpha-1)/\alpha}.$$

Let it be assumed still that $\rho(t)$ has a lower bound $v>0$. Let

$$h_{n2}=\int_{-1}^{1}\rho(t)[H_n(x,t)]^2dt. \tag{7}$$

It has been shown above, in (3) and (5), that h_{n2} does not exceed a constant multiple of n, or in symbols

$$h_{n2}=O(n), \tag{8}$$

this relation holding uniformly for values of x belonging to any closed interval interior to $(-1, 1)$. The integral of H_n^2 from -1 to 1 does not exceed h_{n2}/v, and so, by a lemma analogous to one already cited, but giving an upper bound for the absolute value of the polynomial in the integrand throughout the whole closed interval of integration,*

$$|\,H_n(x,t)\,|=O[n(h_{n2})^{1/2}]$$

uniformly for $-1\leqq t\leqq 1$ and $-1\leqq x\leqq 1$. From this and (8) it appears that $|H_n(x,t)|=O(n^{3/2})$, uniformly for t in $(-1, 1)$ and for x in any closed interval interior to $(-1, 1)$.

For a specified x, let μ_n be the maximum of $|H_n(x,t)|$ in $(-1, 1)$. If $\gamma>2$,

* See this Bulletin, loc. cit., pp. 893–894, Lemma 2. It is to be understood in the statement of the Lemma that $n\geqq 1$.

$$\int_{-1}^{1} \rho(t) \mid H_n(x,t) \mid^{\gamma} dt = \int_{-1}^{1} \mid H_n(x,t) \mid^{\gamma-2} \rho(t) [H_n(x,t)]^2 dt$$

$$\leqq \mu_n^{\gamma-2} \int_{-1}^{1} \rho(t) [H_n(x,t)]^2 dt = \mu_n^{\gamma-2} h_{n2}.$$

Since $\mu_n = O(n^{3/2})$ and $h_{n2} = O(n)$,

$$\int_{-1}^{1} \rho(t) \mid H_n(x,t) \mid^{\gamma} dt = O(n^{(3\gamma/2)-2}). \tag{9}$$

Substitution of this in (6), with $\gamma = \alpha/(\alpha-1)$, gives

$$\left| \int_{-1}^{1} \rho(t)\phi(t) H_n(x,t) dt \right| = O(n^{(3/2)-(2/\gamma)}) = O(n^{(2/\alpha)-(1/2)}); \tag{10}$$

the specification that $\alpha < 2$ insures that $\gamma > 2$. The second integral in (1) can be treated in the same way, and it follows that

$$\mid \sigma_n(x) - f(x) \mid = O(n^{(2/\alpha)-(3/2)}). \tag{11}$$

In any closed interval interior to $(-1, 1)$ the relation holds uniformly over any set of values of x for which $\int \rho \mid \phi \mid^{\alpha}$ is bounded.

The exponent $(2/\alpha)-(3/2)$ will be negative if $\alpha > 4/3$. *If $\rho(t)$ has a positive lower bound, and if x is an interior point of the interval $(-1, 1)$, $\sigma_n(x)$ will converge toward $f(x)$ provided that $\rho(t) \mid \phi(t) \mid^{\alpha}$ is summable with a value of $\alpha > 4/3$, the notation $\phi(t)$ representing the difference quotient $[f(t)-f(x)]/(t-x)$.* Further details with regard to uniformity of convergence can be supplied from the preceding paragraph.

The condition just stated is manifestly broader than that which required that $\rho\phi^2$ be summable. *The condition for convergence of $\sigma_n(x)$ toward $f(x)$ will be satisfied*, as far as the behavior of $f(t)$ in the neighborhood of the point $t = x$ is concerned, if $\rho(t)$ *is bounded in the neighborhood of the point, and if*

$$\mid f(t) - f(x) \mid = O(\mid t - x \mid^{\beta}) \tag{12}$$

as t approaches x, with a value of $\beta > 1/4$. For then

$$\mid \phi(t) \mid = O(\mid t - x \mid^{\beta-1}), \quad \mid \phi(t) \mid^{\alpha} = O(\mid t - x \mid^{(\beta-1)\alpha}),$$

and it will be possible to choose a value of $\alpha > 4/3$ so that $(\beta-1)\alpha > -1$.

4. *More General Weight Function.* A new element is introduced into the problem if the greatest lower bound of ρ, instead of being positive, is allowed to be zero. The extent of this generalization will be limited for the purposes of the present argument by the assumption that $[\rho(t)]^{-r}$ *is summable over* $(-1, 1)$ *with a value of* $r>1$.

The change of hypothesis affects the estimate of an upper bound for $K_n(x, x)$. Let

$$s = 2r/(r+1), \qquad p = 2/s = 1 + (1/r) > 1.$$

By application of Hölder's inequality to the integral of the product of the factors $[\rho(t)]^{-1/p}$ and $[\rho(t)]^{1/p}|K_n(x, t)|^s$, it is seen that*

$$\int_{-1}^{1} |K_n(x, t)|^s dt \leqq \left[\int_{-1}^{1} [\rho(t)]^{-r} dt\right]^{1/(r+1)} \left[\int_{-1}^{1} \rho(t)[K_n(x, t)]^2 dt\right]^{r/(r+1)}.$$

The next to the last integral is convergent by hypothesis, and is independent of x; the value of the last integral is $K_n(x, x)$. By a lemma† which has already been used in a special case,

$$|K_n(x, t)| \leqq \frac{Cn^{(r+1)/(2r)}[K_n(x, x)]^{1/2}}{(1-t^2)^{N/2}}$$

for $n \geqq 1$ and for $-1<t<1$, where N is the smallest integer $\geqq (r+1)/(2r)$, and C is independent of n, x, and t. In particular, for $t=x$,

$$[K_n(x, x)]^{1/2} \leqq \frac{Cn^{(r+1)/(2r)}}{(1-x^2)^{N/2}}, \qquad K_n(x, x) \leqq \frac{C^2 n^{(r+1)/r}}{(1-x^2)^N}.$$

This calculation holds for any value of $r>0$. If $r>1$, $N=1$, and

$$K_n(x, x) \leqq \frac{C^2 n^{(r+1)/r}}{1-x^2}. \tag{13}$$

In continuation, it is to be supposed again that $\rho(t)|\phi(t)|^\alpha$ is summable with a value of α between 1 and 2, and inquiry is to

* See this Bulletin, loc. cit., p. 892.

† This Bulletin, loc. cit., p. 897, Lemma 4.

be made as to a value of α which will insure convergence of $\sigma_n(x)$ toward $f(x)$ under present conditions. The relation (6) holds as before. Also (3) is still valid, and if h_{n2} is defined as in (7), the right-hand member of (13) gives an upper bound for h_{n2}. If $H_n(x, t)/(h_{n2})^{1/2}$ is denoted by $P_n(t)$, it can be asserted* that $|P_n(t)|$ has an upper bound of the order of $n^{(r+1)/r}$ for $-1 \leqq t \leqq 1$; inspection of the proof of the proposition referred to will show that

$$|P_n(t)| \leqq 2^{(3r+1)/(2r)} n^{(r+1)/r} \left[\int_{-1}^{1} [\rho(t)]^{-r} dt\right]^{1/(2r)}$$

for $n \geqq 1$, so that $|P_n(t)| = O(n^{(r+1)/r})$ uniformly for $-1 \leqq t \leqq 1$, $-1 \leqq x \leqq 1$. Consequently $|H_n(x, t)| = O(n^{3(r+1)/(2r)})$ uniformly for $-1 \leqq t \leqq 1$ and for x in any closed interval interior to $(-1, 1)$. The relation (9) becomes

$$\int_{-1}^{1} \rho(t) |H_n(x, t)|^{\gamma} dt = O(n^{(3\gamma-4)(r+1)/(2r)}),$$

with the understanding that $\gamma > 2$, as before, and (10) is to be replaced by

$$\left|\int_{-1}^{1} \rho(t)\phi(t)H_n(x, t)dt\right| = O(n^{(4-\alpha)(r+1)/(2\alpha r)}),$$

the exponent which previously appeared in the right-hand member being multiplied by $(r+1)/r$ in each case. The same order of magnitude is found if $H_n(t, x)$ is substituted for $H_n(x, t)$. The exponent in (11) is replaced by

$$\frac{(4-\alpha)(r+1)}{2\alpha r} - 1,$$

which is negative if $\alpha > (4r+4)/(3r+1)$. The last fraction is greater than 1 for any positive r, and is less than 2 by virtue of the hypothesis that $r > 1$; it is in fact greater than the old value $4/3$, which it approaches when r is very large.

It thus becomes clear how the conditions for convergence at the end of the preceding section are to be modified. The greater generality of the weight function is compensated by a closer

* See this Bulletin, loc. cit., p. 894, Theorem 4.

restriction on the difference quotient $\phi(t)$, to the extent that the lower bound $4/3$ for α is increased to $(4r+4)/(3r+1)$. The corresponding lower bound for β in (12) is $(r+3)/(4r+4)=(1/4)+1/(2r+2)$, instead of $1/4$. This is to be compared with the exponent $(r+1)/(2r)=(1/2)+1/(2r)$ which appears in a theorem obtained elsewhere* for convergence of $s_n(x)$ toward $f(x)$.

5. *More General Summation.* Let the simple arithmetic mean which has been the subject of discussion hitherto be replaced by the more general expression

$$\sigma_n(x) = \sum_{k=0}^{n-1} c_{nk} s_k(x),$$

where the c's, within limits to be specified presently, are arbitrary constants satisfying the condition that

$$\sum_{k=0}^{n-1} c_{nk} = 1$$

for each value of n. By virtue of the last condition,

$$\sigma_n(x) - f(x) = \sum_{k=0}^{n-1} c_{nk}[s_k(x) - f(x)].$$

For each n let A_n be the largest of the numbers $|c_{nk}|$. (Negative values for some of the c's are not excluded.) Repetition of the reasoning of §2, with obvious modifications in detail, shows that if $\rho\phi^2$ is summable and ρ has a positive lower bound,

$$|\sigma_n(x) - f(x)| = O(n^{1/2}A_n).$$

Convergence will take place if $\lim_{n\to\infty} n^{1/2}A_n = 0$. Specifications with regard to uniformity of convergence are the same as before. It is only superficially a further generalization if $\sum c_{nk}$, instead of being equal to 1 for each n, is merely required to approach 1 as a limit. For if $\sum c_{nk} = e_n$ and $e_n \to 1$ the argument just outlined can be applied to $\sigma_n(x)/e_n$, and this expression and $\sigma_n(x)$ converge simultaneously.

* This Bulletin, loc. cit., p. 906, Theorem 16, $m=2$. The difference in exponents in directly significant, in spite of the fact that it is accompanied by other differences in substance and form of statement.

A detailed working out of the results of combining the principle of this section with those of §§3 and 4 may be omitted.

6. *Trigonometric Approximation.* The methods used above can be applied similarly to problems of the summation of series of orthogonal trigonometric sums.* As frequently occurs in such parallel developments, the work is in some respects simpler in the trigonometric problem and in other respects more complicated. By reason of the fact that in one of the lemmas employed a factor n^2 in the polynomial case is replaced by n in the case of trigonometric sums,† the condition $\alpha>4/3$ in §3 is replaced by the simpler condition $\alpha>1$, and the condition $\beta>1/4$ by $\beta>0$. The corresponding requirements in §4 are $\alpha>1+(1/r)$ and $\beta>1/(r+1)$.

Analogous results can be obtained for the case of polynomial approximation if $\rho(t)$ becomes infinite in a suitable manner at the ends of the interval.‡

THE UNIVERSITY OF MINNESOTA

* See D. Jackson, *Orthogonal trigonometric sums*, Annals of Mathematics, (2), vol. 34 (1933), pp. 799–814.

† See this Bulletin, loc. cit., p. 891, Lemma 1, as compared with Lemma 2, previously cited.

‡ See this Bulletin, loc. cit., pp. 895–896, §4.

A METHOD OF GENERATING PART OF ARITHMETIC WITHOUT USE OF INTUITIVE LOGIC

BY W. V. QUINE*

1. *Introduction.* By an *identity* I mean an equation which is a theorem of ordinary arithmetic (and hence true for all values of its variables). What is meant here by "ordinary arithmetic" may be left to the reader, granted the following three reservations:

(a) The equations

(A) $$x = x - (y - y),$$

(B) $$x - (y - z) = z - (y - x),$$

are identities, that is, theorems of ordinary arithmetic.

(b) Inference according to either of the following rules is valid in ordinary arithmetic:

(R) *Substitute any one expression for all occurrences of any variable.*

(R′) *Given $\alpha = \beta$, put α for β anywhere.*

(c) An equation whose members open with the same variable and contain only variables connected by subtraction, without parentheses, is not an identity unless each variable occurs exactly as many times in one member as in the other.

A homogeneous linear identity with rational coefficients is an identity whose members contain only plus and minus signs, parentheses, "0", variables, and rational numerical coefficients. Aside from the prefixture of such coefficients, no multiplicative juxtaposition is admissible. The expressions thus admitted can all be defined in terms of variables, parentheses and the signs of subtraction and equality. These definitions, which have the status of mere conventions of notational abbreviation, will be set forth in §3.

It is the business of this paper to show that all homogeneous

* Society of Fellows, Harvard University. For suggestions in the preparation of this paper I am indebted to E. V. Huntington and T. P. Palmer.

linear identities with rational coefficients, when thus analyzed into subtraction, equality, and variables, can be generated from (A) *and* (B) *by means solely of mechanical substitution according to* (R) *and* (R′).

2. *Generation of Theorems.* (A) and (B) comprise the postulates, and (R) and (R′) the rules of inference, of a calculus whose primitive language contains only variables, the functor of subtraction and the equality sign.* The parentheses represent nothing in the way of an additional primitive idea, but merely constitute a part of the notation of the functor of subtraction. This functor is the notational scheme of writing expressions in the respective blanks of the matrix " −( )". No parentheses are needed enclosing the blank to the left of "−" in this matrix, for, as is easily verified relatively to subtraction or any other binary operation, the condition that all right-hand operands and only right-hands operands are to be parenthesized suffices in any context for the unique determination of the grouping of left-hand operands as well.

Strictly, (A) and (B) should appear as

$$x = x - (y - (y)) \text{ and } x - (y - (z)) = z - (y - (x)).$$

The suppression of parentheses enclosing single letters is an ellipsis; in using (R) and (R′), we are to imagine that single letters immediately to the right of minus signs are enclosed in parentheses, and we are to make such parentheses explicit when a complex is substituted for such a letter.

Let us proceed to the generation of theorems from (A) and (B) by (R) and (R′). The manner of generation is indicated, to the right of each theorem, by literal or numerical reference to

* The primitive dyadic relation of equality might be suppressed in favor of a primitive predicate; if namely "$\mathfrak{B}$" denote the predicate of equality with 0, so that "$\mathfrak{B}\zeta$" is read "ζ vanishes", we can render any equation $\alpha=\beta$ as $\mathfrak{B}\ \alpha-\beta$. (A) and (B) become

$\mathfrak{B}\ \ x-(x-(y-y))$ and $\mathfrak{B}\ \ (x-(y-z))-(z-(y-x))$.

(R) remains unchanged, but (R′) refers now to $\mathfrak{B}\ \alpha-\beta$ instead of $\alpha=\beta$.

Indeed, since in the thus modified calculus all theorems and postulates begin uniformly with "$\mathfrak{B}$", that letter might be dropped entirely; the list of theorems and postulates would then appear as a list of alternative expressions of 0. A system is thus arrived at whose theorems and postulates are expressive of elements of the system itself, as is the case in the logical calculus of propositions.

the postulates or intermediate theorems from which the theorem in question is derived. A single reference "t" means that the generation proceeds by substitution in the theorem or postulate (t) according to (R). A double reference "(s), (t)" means that the generation proceeds by replacement of an occurrence of the right member of (s), within (t), by the left member of (s), on the authority of (R′).

(1)	$x = x,$	(A), (A).
(2)	$x = x - (z - z),$	(A).
(3)	$y - (x - z) = z - (x - y),$	(B).
(4)	$y - (x - (z - z)) = z - z - (x - y),$	(B).
(5)	$z - (z - z - (x - y)) = x - y - (z - z - z),$	(B).
(6)	$y - y - (x - x - x) = y - y - (x - x - x)$	(1).
(7)	$y - x = z - z - (x - y),$	(2), (4).
(8)	$x - z - y = w - w - (y - (x - z)),$	(7).
(9)	$x - y - z = w - w - (z - (x - y))$	(7).
(10)	$z - (y - x) = x - y - (z - z - z),$	(7), (5).
(11)	$x - (y - y) = y - y - (x - x - x),$	(10).
(12)	$x - y - z = w - w - (y - (x - z)),$	(3), (9).
(13)	$x - (y - z) = x - y - (z - z - z),$	(B), (10).
(14)	$x - (y - y) = x - y - (y - y - y),$	(13).
(15)	$x - y - z = x - z - y,$	(8), (12).
(16)	$z - z - (x - y) = z - (x - y) - z,$	(15).
(17)	$y - x = z - (x - y) - z,$	(7), (16).
(18)	$y - x = y - (x - z) - z,$	(3), (17).
(19)	$x = x - y - (y - y - y),$	(A), (14).
(20)	$x = y - y - (x - x - x),$	(A), (11).
(21)	$y - y - (x - x - x) = x,$	(20), (6).

3. *Phrases.* By a *phrase* I shall mean any expression built up of variables and the functor of subtraction. A more rigorous de-

scription is the following: Letters, also the expression "$x-y$", are phrases, and if a letter in a phrase be replaced by a phrase (with restoration of requisite parentheses, in conformity with §2) the result is a phrase. A *phrase-equation* is an equation whose members are phrases; a *phrase-identity* is a phrase-equation which is an identity.

A homogeneous linear identity with rational coefficients is not necessarily a phrase-identity, for, over and above expressions of subtraction, its members may involve expressions of algebraic negation or addition, the sign "0", and rational numerical coefficients. These further devices are all definable, however, as means of abbreviating phrases or phrase-equations. "0" can be explained as an abbreviation for the specific expression "$x-x$". Negation can be accounted for by defining $-z$ in general as $0-z$, that is, by construing "$-(\quad)$" as an abbreviation for "$0-(\quad)$". Addition can be introduced similarly by defining $x+y$ as $x-(-y)$, and the integral numerical coefficients are provided for by defining $2x$ as $x+x$, $3x$ as $2x+x$, and so on.* The symbols "$0x$" and "$1x$" may be taken as notational variants of "0" and "x".

Fractional numerical coefficients, finally, can be defined contextually by the following abbreviative conventions. An equation

$$\text{“}\alpha - \beta - \cdots - \frac{m}{n}\zeta - \cdots - \theta = \omega\text{”},$$

with any disposition of parentheses, is an abbreviation for

$$\text{“}n\alpha - n\beta - \cdots - m\zeta - \cdots - n\theta = n\omega\text{”},$$

with the corresponding disposition of parentheses, where m is any numeral and n any numeral other than "0". The corresponding convention is adopted where the fraction occurs to the right of the equality sign. If several fractional coefficients occur in an equation, their explanation by the above conventions is to proceed in a left-to-right order. This set of conventions provides for the occurrence of fractional coefficients not only in subtraction but in the other contexts as well, when the

* See E. V. Huntington, Annals of Mathematics, (2), vol. 8 (1906), p. 15.

definitions of the preceding paragraph are brought into play.*

Every homogeneous linear identity with rational coefficients is either a phrase-identity or an abbreviation of a phrase-identity by the above abbreviative conventions. The proof that every homogeneous linear identity with rational coefficients is, upon elimination of the above abbreviations, generable by (R) and (R′) from (A) and (B), will therefore consist in showing that every phrase-identity is generable by (R) and (R′) from (A) and (B).

4. *Canonical Form of an Equation.* An equation will be said to be of *canonical form* if each of its members is a phrase in which no parentheses occur and in which like letters appear consecutively, and if the initial letters of the two members are alike and the order of succession of unlike letters is, barring omissions, the same in both members. The canonical form is thus

$$\text{(i)}\quad v_1 - v_1 - v_1 - \cdots - v_2 - v_2 - \cdots - v_n - v_n = v_1 - v_1 - v_1 - \cdots - v_2 - v_2 - \cdots - v_n - v_n$$

where v_i and v_j are unlike letters unless $i=j$, and where, for any m, the number of occurrences of "$-v_m$" in either member may be 0.

In saying that an equation E is *derivable* from an equation F, I shall mean that E is generable by (R) and (R′) from (A), (B), and F. In saying that E is *interderivable* with F, I shall mean that E is derivable from F and F from E. In saying that an equation is *reducible to canonical form*, I shall mean that it is interderivable with an equation of canonical form.

Since the initial letters of the members of an equation of canonical form are alike, §1(c) tells us that an equation (i) of canonical form is not an identity unless each variable v_i occurs the same number of times in one member as in the other. But under such circumstances the members of (i) are exact duplicates of each other, so that the equation is generable by (R) from the principle (1) of self-equality. Since (1) is one of the

* It is seen from (A) and (B) that the structure of the calculus developed in §2 admits of another application, where the elements are non-vanishing numbers and "$x-y$" is re-interpreted as meaning "x divided by y". Under this re-interpretation, 0 as above defined gives way to 1, the negative gives way to the reciprocal, the sum to the product, and the numerical coefficients to the corresponding numerical exponents.

theorems already generated by (R) and (R′) from (A) and (B), it follows that every identity of canonical form is generable by (R) and (R′) from (A) and (B).

By §1(b), any equation generable by (R) and (R′) from theorems of ordinary arithmetic is a theorem of ordinary arithmetic, and hence an identity. In view of §1(a), therefore, any equation generable by (R) and (R′) from (A), (B), and E is an identity if E is an identity. Now let E be an identity reducible to canonical form, that is, interderivable with some equation F of canonical form. Equation F is then an identity, since it is generable by (R) and (R′) from (A), (B), and E. But, being an identity of canonical form, F is generable by (R) and (R′) from (A) and (B) alone. Hence E, derivable as it is from F, is likewise generable by (R) and (R′) from (A) and (B). It is thus established that every identity E which is reducible to canonical form is generable by (R) and (R′) from (A) and (B).

In order to show then that every phrase-identity is generable by (R) and (R′) from (A) and (B), it will be sufficient to prove that all phrase-equations (and hence in particular all phrase-identities) are reducible to canonical form. This proof occupies the remaining section.

5. *Proof of Reducibility.* Given any equation

(ii) $\alpha = \beta,$

the reverse equation (iv) is derivable as follows:

(iii) $\beta = \beta,$ (1),

(iv) $\beta = \alpha,$ (ii), (iii).

Hence the following principle is established.

I. *Any equation $\alpha = \beta$ is interderivable with its reverse $\beta = \alpha$.*

By the *terms* of a phrase α I mean all letters of α not lying within parentheses in α, and all parenthesized phrases in α not lying within any broader parenthesis in α. Since parentheses can open only to the right of a minus sign [see §2], the initial term of a phrase is necessarily a letter.

From any phrase-equation

(v) $x - \alpha_1 - \alpha_2 - \cdots - \alpha_i - \alpha_{i+1} - \cdots - \alpha_n = \omega,$

we can derive an equation (vii) as follows:

(vi) $x - \alpha_1 - \cdots - \alpha_{i-1} - \alpha_{i+1} - \alpha_i$
$= x - \alpha_1 - \cdots - \alpha_{i-1} - \alpha_i - \alpha_{i+1}$, (15),

(vii) $x - \alpha_1 - \cdots - \alpha_{i-1} - \alpha_{i+1}$
$- \alpha_i - \alpha_{i+2} - \cdots - \alpha_n = \omega$, (vi), (v).

By the same process, with α_i and α_{i+1} interchanged, we can derive (v) from (vii). Thus any phrase-equation (v) is interderivable with an equation (vii) resulting from the permutation of any non-initial term of the left member of (v) with its successor. Serial application of this principle establishes the interderivability of (v) with the equation resulting from *any* permutation of non-initial terms in the left member of (v). In view of I, furthermore, the same will be true relatively to the right member. Hence the following principle is inferred.

II. *Any phrase-equation E is interderivable with any equation resulting from permutation of non-initial terms within either member of E.*

From any phrase-equation

(viii) $y - \alpha_1 - \alpha_2 - \cdots - \alpha_m = x - \beta_1 - \beta_2 - \cdots - \beta_n$,

we can derive an equation (ix) as follows:

(ix) $y - \alpha_1 - \cdots - \alpha_m$
$= y - y - (x - x - x) - \beta_1 - \cdots - \beta_n$, (21), (viii).

Conversely, (viii) is derivable from (ix) as follows:

[viii] $y - \alpha_1 - \cdots - \alpha_m = x - \beta_1 - \cdots - \beta_n$, (20), (ix).

Therefore (viii) is interderivable with (ix). Hence the following principle is established.

III. *Any phrase-equation is interderivable with a phrase-equation, the initial letters of whose members are alike.*

Since parentheses can occur in a phrase only by way of enclosing a non-initial term thereof, it follows from II that if any parenthesized expression in a phrase-equation be transported to the end of the member in which it occurs, the resulting equation will be interderivable with the original one. Thus, in particular, any phrase-equation

(x) $x - \alpha_1 - \alpha_2 - \cdots - \alpha_m$
$= x - \beta_1 - \beta_2 - \cdots - \beta_{i-1} - (\zeta - \theta) - \beta_{i+1} - \cdots - \beta_n$,

whose members have like initial letters and whose right member contains parentheses as indicated, is interderivable with

(xi) $x - \alpha_1 - \cdots - \alpha_m$
$= x - \beta_1 - \cdots - \beta_{i-1} - \beta_{i+1} - \cdots - \beta_n - (\zeta - \theta)$.

Now from (xi) we can derive an equation (xiii) as follows:

(xii) $x - \beta_1 - \cdots - \beta_{i-1} - \beta_{i+1} - \cdots - \beta_n - \zeta$
$= x - \beta_1 - \cdots - \beta_{i-1} - \beta_{i+1} - \cdots - \beta_n$
$- (\zeta - \theta) - \theta$, (18),

(xiii) $x - \beta_1 - \cdots - \beta_{i-1} - \beta_{i+1} - \cdots - \beta_n - \zeta$
$= x - \alpha_1 - \cdots - \alpha_m - \theta$, (xi), (xii).

Conversely, (xi) is derivable from (xiii) as follows:

(xiv) $x - \alpha_1 - \cdots - \alpha_m = x - \alpha_1 - \cdots - \alpha_m$
$- \theta - (\theta - \theta - \theta)$, (19),

(xv) $x - \alpha_1 - \cdots - \alpha_m = x - \beta_1 - \cdots - \beta_{i-1} - \beta_{i+1}$
$- \cdots - \beta_n - \zeta - (\theta - \theta - \theta)$, (xiii), (xiv),

(xvi) $x - \beta_1 - \cdots - \beta_{i-1} - \beta_{i+1} - \cdots - \beta_n - (\zeta - \theta)$
$= x - \beta_1 - \cdots - \beta_{i-1} - \beta_{i+1} - \cdots - \beta_n$
$- \zeta - (\theta - \theta - \theta)$, (13),

(xi) $x - \alpha_1 - \cdots - \alpha_m = x - \beta_1 - \cdots - \beta_{i-1}$
$- \beta_{i+1} - \cdots - \beta_n - (\zeta - \theta)$, (xvi), (xv).

Thus (xi), and hence also (x), is interderivable with (xiii). By I, then, the reverse of (x) is likewise interderivable with (xiii). But in the reverse of (x), $(\zeta - \theta)$ occurs in the left member rather than the right. Hence, in conclusion, a phrase-equation whose members have initial letters and either of whose members contains a parenthesized expression, is interderivable with a phrase-equation (xiii) in which the members still have like initial letters but in which the parenthesized expression is broken up. Serial application of this principle establishes the following one.

IV. *Any phrase-equation whose members have like initial letters is interderivable with a phrase-equation whose members are free of parentheses and have like initial letters.*

From the definition of the "terms" of a phrase, it is clear that if a phrase is free of parentheses all its terms are letters. Hence, by II, any phrase-equation E whose members are free of parentheses is interderivable with any phrase-equation resulting from permutation of non-initial letters within members of E. Therefore any phrase-equation whose members are free of parentheses and have like initial letters is interderivable with an equation of canonical form. It then follows, by III and IV, that every phrase-equation is reducible to canonical form.

In view of §§3–4, this concludes the proof that upon elimination of abbreviations all homogeneous linear identities with rational coefficients are generable by (R) and (R′) from (A) and (B).

HARVARD UNIVERSITY

A CHARACTERISTIC PROPERTY OF SURFACES OF NEGATIVE CURVATURE†

BY E. F. BECKENBACH

1. *Introduction.* Let there be given a piece of surface S in a representation

(1) S: $x = x(u, v),\ y = y(u, v),\ z = z(u, v),\ u^2 + v^2 < \rho^2,$

with the following properties.

(a) $x(u, v), y(u, v), z(u, v)$ have continuous partial derivatives of the third order.

(b) The representation is isothermic; that is to say, $E = G$, $F = 0$, where

$$E = x_u^2 + y_u^2 + z_u^2, F = x_u x_v + y_u y_v + z_u z_v, G = x_v^2 + y_v^2 + z_v^2,$$

the subscripts denoting differentiation.

We put $E = G = \lambda(u, v)$. Then $\lambda(u, v) \geqq 0$, and the representation is conformal except at points where $\lambda(u, v) = 0$. The cus-

† Presented to the Society, April 7, 1934.

tomary assumption of differential geometry, that $\lambda(u, v)>0$, is dropped because that assumption seems irrelevant to the present results.

Let the circle $(u-u_0)^2+(v-v_0)^2\leqq r^2$ in $u^2+v^2<\rho^2$ be denoted by $(u_0, v_0; r)$; and let the area and length of boundary of the image on S of $(u_0, v_0; r)$ be denoted respectively by $a(u_0, v_0; r)$ and $l(u_0, v_0; r)$. On the above assumptions, the theorems expressed by the following formulas were proved first if S lies on a plane,† then if S is a minimal surface,‡ and finally if S is a surface of non-positive Gaussian curvature K:§

$$a(u_0, v_0; r) \geqq \pi r^2\lambda(u_0, v_0), \tag{2}$$

$$l(u_0, v_0; r) \geqq 2\pi r[\lambda(u_0, v_0)]^{1/2}. \tag{3}$$

Here it will be noted that $\lambda(u_0, v_0)$ and $[\lambda(u_0, v_0)]^{1/2}$ are respectively the area magnification ratio and the length magnification ratio at (u_0, v_0).

Any representation of a (u, v)-domain D on S will be called a *typical* map provided conditions (a) and (b) are satisfied for that representation.

It is our purpose to demonstrate the extent to which these formulas (2) and (3) are *characteristic* of surfaces of negative Gaussian curvature K, namely, we shall show that (see §3) there exist typical maps of (u, v)-domains D on surfaces of positive curvature for which the above inequalities are satisfied for all $(u_0, v_0; r)$ in D; but that (see §6) if either of these inequalities holds for *all* typical maps of (u, v)-domains D on S and for all $(u_0, v_0; r)$ in D, then $K\leqq 0$ on S.‖

The above will follow immediately from the simple lemma (see §§4, 5): *A necessary and sufficient condition that the Gaussian curvature K of a surface S, which admits of typical representations, be $\leqq 0$ wherever K is defined on S, is that for all typical maps of (u, v)-domains on S, the function $\lambda(u, v)$ be subharmonic.*

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† L. Bieberbach, *Zur Theorie und Praxis der konformen Abbildung*, Palermo Rendiconti, vol. 38 (1914), pp. 98–112.

‡ E. F. Beckenbach, *The area and boundary of minimal surfaces*, Annals of Mathematics, vol. 33 (1932), pp. 658–664.

§ E. F. Beckenbach and T. Radó, *Subharmonic functions and surfaces of negative curvature*, Transactions of this Society, vol. 35 (1933), pp. 662–674.

‖ That is, $K\leqq 0$ wherever K is defined on S, namely at all points for which $\lambda(u, v)>0$.

2. *Lemma.* Let S be a surface given in a representation

$$S: \quad x = x(u, v),\ y = y(u, v),\ z = z(u, v),$$

(u, v) in some domain D (not necessarily simply-connected), the coordinate functions having continuous first partial derivatives satisfying $E = G = \lambda(u, v)$, $F = 0$. If $\lambda(u, v)$ is subharmonic in D, then $a(u_0, v_0; r)/(\pi r^2)$ is a convex function of $\log r$ in any circular disc lying in D. Or if $[\lambda(u, v)]^{1/2}$ is subharmonic in D, then† $l(u_0, v_0; r)/(2\pi r)$ is a convex function of $\log r$ in any circular annulus lying in D.

This lemma merely expresses special cases of a fundamental property of the linear-mean and of the area-mean of subharmonic functions.‡

Thus

$$\frac{l(u_0, v_0; r)}{2\pi r} = \frac{1}{2\pi}\int_0^{2\pi} [\lambda(u_0 + r\cos\phi, v_0 + r\sin\phi)]^{1/2} d\phi$$

is the linear-mean of $[\lambda(u, v)]^{1/2}$, and

$$\frac{a(u_0, v_0; r)}{\pi r^2} = \frac{1}{\pi r^2}\iint_{\xi^2+\eta^2<r^2} \lambda(u_0 + \xi, v_0 + \eta)\, d\xi\, d\eta$$

is the area-mean of $\lambda(u, v)$.

For any point (u_0, v_0) of D, we define, for $r = 0$,

$$\left[\frac{a(u_0, v_0; r)}{\pi r^2}\right]_{r=0} = \lim_{r=0} \frac{a(u_0, v_0; r)}{\pi r^2} = \lambda(u_0, v_0),$$

$$\left[\frac{l(u_0, v_0; r)}{2\pi r}\right]_{r=0} = \lim_{r=0} \frac{l(u_0, v_0; r)}{2\pi r} = [\lambda(u_0, v_0)]^{1/2}.$$

We have then the immediate corollaries, which hold as long as the circular disc $(u_0, v_0; r)$ remains in D:

† See C. H. Dix, *The length of closed level curves of a harmonic function*, this Bulletin, vol. 39 (1933), pp. 24–25.

‡ F. Riesz, *Über subharmonische Funktionen und ihre Rolle in der Funktionentheorie und in der Potentialtheorie*, Acta Szeged, vol. 2 (1924–26), pp. 87–100, and *Sur les fonctions subharmoniques et leur rapport à la théorie du potentiel*, Acta Mathematica, vol. 48 (1926), pp. 329–343. For the generalization to the area-mean, see P. Montel, *Sur les fonctions convexes et les fonctions sousharmoniques*, Journal de Mathématiques, (9), vol. 7 (1928), pp. 29–60.

If $\lambda(u, v)$ is subharmonic in D, then

(A.1) $$a(u_0, v_0; r)/(\pi r^2)$$

is a non-decreasing function of r;

(A.2) $$a(u_0, v_0; r) \geqq \pi r^2 \lambda(u_0, v_0).$$

If $[\lambda(u, v)]^{1/2}$ is subharmonic in D, then

(B.1) $$l(u_0, v_0; r)/(2\pi r)$$

is a non-decreasing function of r;†

(B.2) $$l(u_0, v_0; r) \geqq 2\pi r[\lambda(u_0, v_0)]^{1/2}.$$

(A.1) and (B.1) follow from the fact that a function which is convex out to $-\infty$, and is bounded, necessarily is non-decreasing. (A.2) and (B.2) are immediate corollaries of (A.1) and (B.1), respectively.

If in the above we replace the word "subharmonic" by "superharmonic," the lemma still holds, with "convex" replaced by "concave." And the four corollaries hold, with "non-decreasing" replaced by "non-increasing" and with the signs of inequality reversed.

3. *Example.* The equations‡

$$x = \frac{2u}{u^2 + v^2 + 1}, \quad y = \frac{2v}{u^2 + v^2 + 1}, \quad z = \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}$$

give the familiar stereographic projection of the unit sphere ($K \equiv +1$) with center (0, 0, 0) in the (x, y, z)-space on the (u, v)-plane, the u- and v-axes coinciding, respectively, with those of x and y.

As is well known, this representation is isothermic; indeed, computations yield

$$E = G = \lambda(u, v) = \frac{4}{(u^2 + v^2 + 1)^2}, \quad F = 0.$$

Further computations give the following expressions for the Laplacians $\Delta\lambda \equiv \lambda_{uu} + \lambda_{vv}$ and $\Delta(\lambda^{1/2}) \equiv (\lambda^{1/2})_{uu} + (\lambda^{1/2})_{vv}$:

† A fortiori, then, $l(u_0, v_0; r)$ is a non-decreasing function of r; see E. F. Beckenbach and T. Radó, loc. cit.

‡ See, for instance, L. R. Ford, *Automorphic Functions*, 1929, p. 120.

$$\Delta\lambda = \frac{32}{(u^2 + v^2 + 1)^4}(2u^2 + 2v^2 - 1),$$

$$\Delta(\lambda^{1/2}) = \frac{8}{(u^2 + v^2 + 1)^3}(u^2 + v^2 - 1).$$

Since a necessary and sufficient condition that a function $g(u, v)$ which admits of continuous second derivatives be subharmonic (superharmonic) is that Δg be $\geqq 0$ ($\leqq 0$),† it follows that $\lambda(u, v)$ is subharmonic for $u^2+v^2>1/2$ and superharmonic for $u^2+v^2<1/2$, and that $[\lambda(u, v)]^{1/2}$ is subharmonic for $u^2+v^2>1$ and superharmonic for $u^2+v^2<1$.

It follows (see §2) that (2) holds provided the circle $(u_0, v_0; r)$ lies with its interior in $u^2+v^2\geqq 1/2$, but that the sign of inequality is reversed if the circle lies in $u^2+v^2\leqq 1/2$. A similar statement holds for (3), according as the circle lies with its interior in $u^2+v^2\geqq 1$ or in $u^2+v^2\leqq 1$.

Thus, whether or not $\lambda(u, v)$ for a small piece S^* of the spherical surface is subharmonic, and whether or not (2) or (3) holds on S^*, depend not solely on the nature of the surface but also, in this example, on the choice of typical parameters.

4. THEOREM 1. *If $K\leqq 0$ wherever K is defined on a surface S which admits of typical representations, then for any typical representation of a (u, v)-domain on S, the function $\lambda(u, v)$ is of class PL,‡ and therefore $[\lambda(u, v)]^{1/2}$ and $\lambda(u, v)$ are subharmonic.*

For surfaces in typical representation, $E=G=\lambda(u, v)$, $F=0$, the Gaussian curvature K is given, at points where K is defined, by the expression:§

$$(4) \qquad K = \frac{1}{2\lambda^3}(\lambda_u^2 + \lambda_v^2 - \lambda\Delta\lambda) = -\frac{1}{2\lambda}\Delta \log \lambda.$$

† See F. Riesz, loc. cit.

‡ A function $p(u, v)$, defined in a domain D, is said to be of class PL in D provided $p(u, v)$ is continuous and $\geqq 0$ in D, and $\log p(u, v)$ is subharmonic in the part of D where $p(u, v)>0$. See E. F. Beckenbach and T. Radó, *Subharmonic functions and minimal surfaces*, Transactions of this Society, vol. 35 (1933), pp. 648–661.

§ See, for instance, E. F. Beckenbach and T. Radó, loc. cit.; the results of this section are taken from that source.

It is seen here that $K \leqq 0$ is equivalent to $\Delta \log \lambda \geqq 0$; consequently, $\lambda(u, v)$ is of class PL, and therefore, a fortiori, $[\lambda(u, v)]^{1/2}$ and $\lambda(u, v)$ are subharmonic. Or directly, if it is desired to show only that $[\lambda(u, v)]^{1/2}$ is subharmonic, we note first that at points (u_0, v_0) where $\lambda = 0$, it follows from $\lambda^{1/2} \geqq 0$, that $\lambda^{1/2}$ satisfies the sub-mean property

$$[\lambda(u_0, v_0)]^{1/2} \leqq \frac{1}{2\pi} \int_0^{2\pi} [\lambda(u_0 + r \cos \phi, v_0 + r \sin \phi)]^{1/2} d\phi,$$

and secondly that for other points, $K \leqq 0$ and (4) imply

$$\Delta(\lambda^{1/2}) = \tfrac{1}{2}\lambda^{-3/2}[\lambda \Delta\lambda - \tfrac{1}{2}(\lambda_u^2 + \lambda_v^2)] \geqq 0;$$

consequently, $[\lambda(u, v)]^{1/2}$ is subharmonic. A similar direct discussion might be given for $\lambda(u, v)$.

It follows (see §2) that for any typical representation of a (u, v)-domain D on a surface S for which $K \leqq 0$ wherever K is defined on S, the inequalities (2) and (3) hold for all $(u_0, v_0; r)$ in D.

Conversely, (4) shows that if $\lambda(u, v)$ is of class PL for one typical representation on S, then $K \leqq 0$ wherever K is defined on S; consequently, $\lambda(u, v)$ is of class PL, and therefore subharmonic, for all typical representations of (u, v)-domains on S.

The example of §3 shows, on the other hand, that $\lambda(u, v)$ being subharmonic for one typical representation on a surface S does not imply $K \leqq 0$ on S. Nevertheless, we have the following result.

5. THEOREM 2. *If for all typical representations of (u, v)-domains on the surface S given by* (1), *the area magnification ratio $\lambda(u, v)$ is a subharmonic function of u and v, then the Gaussian curvature K of S is $\leqq 0$ wherever K is defined on S.*

With S given in the representation (1), fix (u_0, v_0), any point in $u^2 + v^2 < \rho^2$; we shall show, on the above assumption, that $K \leqq 0$ at the point on S corresponding to (u_0, v_0), provided K is defined there. Since $u^2 + v^2 < \rho^2$ can be mapped conformally on itself, any interior point being carried to the origin, and since this mapping induces a new typical representation of $u^2 + v^2 < \rho^2$ on S, there is no loss of generality in assuming $(u_0, v_0) = (0, 0)$. Again, since a rotation about the origin gives a conformal map

of the circle on itself, we may assume that the v-axis is tangent, at (0, 0), to a contour line for the function $\lambda(u, v)$, so that

$$\lambda_v(0, 0) = 0. \tag{5}$$

Let an (α, β)-domain D be mapped conformally on $u^2+v^2<\rho^2$ by the analytic function

$$u + iv = f(\alpha + i\beta). \tag{6}$$

This mapping induces a typical representation of D on S, for which the area magnification ratio is given by

$$\mu(\alpha, \beta) = x_\alpha^2 + y_\alpha^2 + z_\alpha^2 = \lambda(u, v)\,|\,f'(\alpha + i\beta)\,|^2.$$

In (6) we substitute the particular function

$$u + iv = (\alpha + i\beta)^m - 2\rho, \tag{7}$$

where m is a real parameter different from zero. This function maps the angle $|\phi| < \pi/(2|m|)$, where $\alpha+i\beta=re^{i\phi}$, on the right-hand half-plane $u>-2\rho$, and consequently maps some domain D in the angle conformally on the circle $u^2+v^2<\rho^2$. Of course, D might not lie on a single-sheeted plane, for D might wind about the origin; but D remains away from the origin and is free of branch-points.

For the function (7), we have

$$u = r^m \cos m\phi - 2\rho, \quad v = r^m \sin m\phi,$$
$$\mu(\alpha, \beta) = \lambda(u, v)m^2r^{2m-2}.$$

A computation shows that the Laplacian

$$\Delta\mu = \mu_{rr} + \frac{\mu_r}{r} + \frac{\mu_{\phi\phi}}{r^2}$$

is given by

$$\Delta\mu = m^4r^{2m-4}\left\{4\lambda\left(\frac{m-1}{m}\right)^2 + 4\lambda_u r^m\left(\frac{m-1}{m}\right)\cos m\phi + 4\lambda_v r^m\left(\frac{m-1}{m}\right)\sin m\phi + r^{2m}\Delta\lambda\right\}.$$

By assumption, $\Delta\mu \geqq 0$ for all $m \neq 0$ and for all (α, β) in D, so that the quantity in brackets is $\geqq 0$ for (u, v) in $u^2+v^2<\rho^2$. In particular, at $(u, v) = (0, 0)$, we have $\phi = 0$ and $r^m = 2\rho$, so

that, writing k for $(m-1)/m$, we get at $(u, v)=(0, 0)$ the quadratic inequality in k

$$\lambda(0, 0)k^2 + 2\lambda_u(0, 0)\rho k + \rho^2\Delta\lambda(0, 0) \geqq 0$$

for all $k \neq 1$, and therefore for all k. Necessary conditions for this are that $\Delta\lambda(0, 0) \geqq 0$ and that the discriminant of the quadratic be $\leqq 0$:

$$[\lambda_u(0, 0)]^2 - \lambda(0, 0)\Delta\lambda(0, 0) \leqq 0. \tag{8}$$

It follows from (4), (5) and (8) that if $\lambda(0, 0) > 0$, so that K is defined at the point on S corresponding to $(u, v)=(0, 0)$, then $K \leqq 0$ there.

6. *Conclusion.* Inequalities (2) and (3), written as

$$\lambda(u_0, v_0) \leqq \frac{1}{\pi r^2}\iint_{\xi^2+\eta^2<r^2} \lambda(u_0+\xi, v_0+\eta)d\xi\, d\eta,$$

$$[\lambda(u_0, v_0)]^{1/2} \leqq \frac{1}{2\pi}\int_0^{2\pi} [\lambda(u_0 + r\cos\phi, v_0 + r\sin\phi)]^{1/2}d\phi,$$

express defining properties* of subharmonic functions. Consequently if, for a certain typical representation on a surface S of a (u, v)-domain D, (2) or (3) holds for all $(u_0, v_0; r)$ in D, then for that representation $\lambda(u, v)$ or $[\lambda(u, v)]^{1/2}$, as the case might be, is subharmonic. In either case, $\lambda(u, v)$ is subharmonic for that representation, since $[\lambda(u, v)]^{1/2}$ being subharmonic implies the same for $\lambda(u, v)$.

By the result of §5, then, if either (2) or (3) holds for *all* typical representations on S of (u, v)-domains D and for all $(u_0, v_0; r)$ in D, so that $\lambda(u, v)$ is subharmonic for all typical representations, it follows that $K \leqq 0$ wherever K is defined on S. And conversely, as we have pointed out (see §4), if S is a surface given in typical representation, for which $K \leqq 0$ wherever K is defined on S, then (2) and (3) hold for all $(u_0, v_0; r)$ in the domain of definition. To this extent, then, the inequalities (2) and (3) are characteristic of surfaces of negative curvature.

THE RICE INSTITUTE

* See J. E. Littlewood, *On the definition of a subharmonic function*, London Mathematical Society Journal, vol. 2 (1927), pp. 189–192.

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

NEW YORK CITY, October 27, 1934.

Abstracts must be in the hands of Associate Secretary J. R. Kline, 501 West 116th St., New York City, not later than September 29. Abstracts received by the Secretaries by September 7 will appear in the September issue of this Bulletin. By invitation of the Program Committee, Professor Emmy Noether, visiting lecturer at Bryn Mawr College, will deliver an address entitled *Modern hypercomplex series.*

LINCOLN, NEBRASKA, November 30-December 1, 1934.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than November 3. All these will appear in the November issue of this Bulletin. By invitation of the Program Committee, addresses will be delivered by Professor G. A. Bliss on *The Problems of Bolza in the calculus of variations,* on *Complex roots of Algebraic equations* and by Professor A. J. Kempner.

UNIVERSITY OF CALIFORNIA AT LOS ANGELES, December 1, 1934.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, University of California, Berkeley, California, not later than November 3. All these will appear in the November issue of this Bulletin.

PITTSBURGH, PENNSYLVANIA, Annual Meeting, December 27-31, 1934.

This meeting is to be held in conjunction with Section A of the A.A.A.S. and the M.A.A. Professor A. B. Coble will deliver his retiring presidential address. There will be a joint session with the American Physical Society, at which time there will be a symposium on mathematical physics with special reference to quantum mechanics. Professor Albert Einstein will deliver the eleventh Josiah Willard Gibbs Lecture. The Cole Prize in Algebra will be awarded. At a joint session with Section A of the A.A.A.S. and the Mathematical Association, addresses will be delivered by Professor C. N. Moore, as retiring Chairman of Section A, on *Mathematics and Science,* and by Professor Arnold Dresden as retiring President of the Mathematical Association.

Abstracts must be in the hands of Associate Secretary J. R. Kline, 501 West 116th St., New York City, not later than November 22, 1934. Abstracts received by the Secretaries by November 9 will appear in the November issue of this Bulletin.

NEW YORK CITY, February 23, 1935.

Abstracts will be due on or before January 26, 1935.

NEW YORK CITY, April 19-20, 1935.

Abstracts will be due on or before March 23, 1935.

CHICAGO, ILLINOIS, April 19-20, 1935.

Abstracts will be due on or before March 23, 1935.

STANFORD UNIVERSITY, April 6, 1935.

Abstracts will be due on or before March 9, 1935.

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

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The initiation fees and the annual dues of members of the Society (see this BULLETIN, p. 322, May, 1930; and the List of Officers and Members, September, 1932, p. 66), are payable to the Treasurer of the Society, Professor G. W. Mullins, 501 West 116th St., New York City.

WHOLE NUMBER 421

CONTENTS

PAGE

Some Recent Contributions to Algebraic Geometry. By VIRGIL SNYDER 673

Note on the Iteration of Functions of One Variable. By MORGAN WARD 688

A New Proof of a Theorem of Denjoy, Young, and Saks. By E. H. HANSON 691

Note on the Use of Fractional Integration of Bessel Functions. By H. P. THIELMAN 695

Note concerning Group Postulates. By RAYMOND GARVER 698

Linear Functional Operations on Functions having Discontinuities of the First Kind. By H. S. KALTENBORN 702

Magic Circles. By C. G. P. KUSCHKE 709

The Converse of Waring's Problem. By L. E. DICKSON .. 711

Hypertranscendental Extensions of Partial Differential Fields. By H. W. RAUDENBUSH, JR. 714

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